Randomised Algorithms Supervision 4 Spectral graph theory: Solution Notes

1 Linear Algebra

Exercise 1 [Properties of dagger] Prove the following properties of the conjugate transpose, defined for any $x \in \mathbb{C}^n$ as

$$x^\dagger = (x_1^*, \ldots, x_n^*).$$

(a) For any two vectors $x, y \in \mathbb{C}^n$, it holds that $x^\dagger y = (y^\dagger x)^\dagger$.

(b) For any vector $x \in \mathbb{C}^n$, it holds that $x^\dagger x \geq 0$.

(*Answer*)

(a) By writing out the definition of the two vectors, we have that

$$(y^\dagger x)^\dagger = (y^\dagger x)^* = \left(\sum_{i=1}^n y_i^* x_i\right)^* = \sum_{i=1}^n (y_i^* x_i)^* = \sum_{i=1}^n y_i x_i^* = x^\dagger y.$$

(b) For any vector $x \in \mathbb{C}^n$,

$$x^\dagger x = \sum_{i=1}^n x_i^* x_i = \sum_{i=1}^n |x_i|^2 \geq 0.$$

Extended Note 1 A matrix $A \in \mathbb{C}^{n \times n}$ is *Hermitian* if $A^\dagger = A$. The notation $A^\dagger$ means the complex conjugate of A, i.e., $A^\dagger = (A^T)^*$. If you prefer you can attempt the following exercises, by assuming that A is a matrix with real entries and then $A^\dagger = A^T = A$, meaning that the matrix is *symmetric*.

Exercise 2 [Real eigenvalues] Consider a Hermitian matrix $A \in \mathbb{R}^{n \times n}$, then all its eigenvalues are real.

(*Answer*) Let A be a Hermitian matrix. Then, for any vector $x \in \mathbb{C}^n$, it holds that

$$(Ax)^\dagger x = x^\dagger A^\dagger x = x^\dagger (Ax). \tag{1}$$

Assuming that x is an eigenvector corresponding to eigenvalue λ, i.e., $Ax = \lambda x$. Then, we have that

$$(Ax)^\dagger x = (\lambda x)^\dagger x = \lambda^* x^\dagger x$$

and

$$x^\dagger (Ax) = x^\dagger \lambda x = \lambda x^\dagger x$$

By 1, we have that

$$\lambda x^\dagger x = \lambda^* x^\dagger x \Rightarrow (\lambda - \lambda^*) x^\dagger x = 0.$$

Hence, since $x^\dagger x > 0$ (as $x \neq 0$), we have that $\lambda = \lambda^*$ and hence λ is real.

Exercise 3 [Orthogonal eigenvectors] Consider a Hermitian matrix $A \in \mathbb{C}^{n \times n}$ and let $x, y \in \mathbb{C}^n$ be two eigenvectors corresponding to different eigenvalues $\lambda \neq \lambda'$. Then, $x \perp y$.

(*Answer*) Since A is Hermitian, we have that

$$(Ax)^\dagger y = x^\dagger A^\dagger y = x^\dagger A y = x^\dagger (Ay).$$

Exercise 4 [Spectral theorem] Let $A \in \mathbb{R}^{n\times n}$ be a symmetric matrix.

(a) Let $k \leq n-1$ and let $x_1, \ldots, x_k$ be orthogonal eigenvectors of A. Then, there exists an eigenvector x_{k+1} that is orthogonal to $x_1, \ldots, x_k$.

(b) Prove the spectral theorem,

(c) Argue that A can be written as XDX^{-1} for some matrix X and a diagonal matrix D.

Exercise 5 [Inverse matrix] Let $A \in \mathbb{R}^{n\times n}$ be a symmetric matrix with (orthonormal) eigenvectors $x_1, \ldots, x_n$ corresponding to eigenvalues $\lambda_1, \ldots, \lambda_n$. Prove that:

(a) Let $x \in \mathbb{R}^n$ be arbitrary. By writing $x = (x^T x_1)x_1 + \ldots + (x^T x_n)x_n$, show that

$$x_1 x_1^T + \ldots x_n x_n^T = I.$$

(b) By writing $Ax = AIx$, show that

$$A = \lambda_1 x_1 x_1^T + \ldots \lambda_n x_n x_n^T.$$

(c) Show that if $\lambda_1 \neq 0, \ldots, \lambda_n \neq 0$, then

$$A^{-1} = \frac{1}{\lambda_1} x_1 x_1^T + \ldots + \frac{1}{\lambda_n} x_n x_n^T.$$

Exercise 6 [Power of a matrix] Consider a real symmetric matrix $A \in \mathbb{R}^{n\times n}$ with eigenvectors $x_1, \ldots, x_n$ and eigenvalues $\lambda_1, \ldots, \lambda_n$. Then, for any $k \in \mathbb{N}_{\geq 1}$, the eigenvectors of A^k are $x_1, \ldots, x_n$ and the eigenvalues $\lambda_1^k, \ldots, \lambda_n^k$.

Exercise 7 [Trace of a matrix] The *trace of a matrix* is defined as

$$\mathrm{tr}(A) = \sum_{i=1}^{n} A_{ii}.$$

(a) Show that for any two matrices X and Y, we have that $\mathrm{tr}(XY) = \mathrm{tr}(YX)$.

(b) Show that for any real symmetric matrix A, $\mathrm{tr}(A) = \sum_{i=1}^{n} \lambda_i$.

(*Answer*)

(a) Using the definition of the trace, we have that

$$\mathrm{tr}(XY) = \sum_{i=1}^{n} (XY)_{ii} = \sum_{i=1}^{n}\sum_{j=1}^{n} X_{ij}Y_{ji} = \sum_{j=1}^{n}\sum_{i=1}^{n} X_{ij}Y_{ji} = \sum_{j=1}^{n}\sum_{i=1}^{n} Y_{ji}X_{ij} = \mathrm{tr}(YX).$$

(b) By writing $A = XDX^{-1}$, we have that

$$\mathrm{tr}(XDX^{-1}) = \mathrm{tr}(XX^{-1}D) = \mathrm{tr}(ID) = \mathrm{tr}(D) = \sum_{i=1}^{n} \lambda_i.$$

Exercise 8 [Determinant of symmetric matrix] Consider a real symmetric matrix A. Show that

$$\det(A) = \prod_{i=1}^{n} \lambda_i.$$

Hint: Use the property that $\det(AB) = \det(A) \cdot \det(B)$.

(*Answer*) By writing $A = XDX^{-1}$ we have that

$$\det(A) = \det(XDX^{-1}) = \det(X)\det(D)\det(X^{-1}) = \det(D) = \prod_{i=1}^{n} \lambda_i,$$

using that $\det(X^{-1}) = (\det(X))^{-1}$ and that the determinant of a diagonal matrix is just the product of the entries of the diagonal.

2 Graph matrices

2.1 Adjacency matrix

Exercise 9 [Basic properties] Consider the adjacency matrix of an undirected graph.

(a) Show that $\deg(v_i) = \sum_{j=1}^{n} A_{ij}$.

(b) Show that the adjacency matrix of a d-regular graph has eigenvalue d.

Exercise 10 [Counting paths] Consider the adjacency matrix A of a graph G.

(a) Show that if $(A^k)_{ij} > 0$ for some integer $k > 0$ then there is a path of length k connecting i and j.

(b) Show that $(A^k)_{ij}$ also gives the number of paths connecting i and j with k hops.

(c) Interpret $\mathrm{tr}(A^k) > 0$.

Exercise 11 [Bipartite graphs] Show that for any bipartite graph G with adjacency matrix A, if $\lambda > 0$ is an eigenvalue then $-\lambda$ is also an eigenvalue.

(*Answer*) Let G be a bipartite graph where the left part has k vertices and the right part has $n-k$. We re-index the n nodes of the graph such that the vertices of the left part are $1, \dots k$ and the vertices of the right part are $k+1, \dots, n$. Then, the adjacency matrix can be written as

$$A = \begin{bmatrix} 0 & B \\ B^T & 0 \end{bmatrix}.$$

Let $\begin{bmatrix} x \\ y \end{bmatrix}$ be the eigenvector of A with eigenvalue λ, then

$$A \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 & B \\ B^T & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} Bx \\ B^T y \end{bmatrix} = \lambda \begin{bmatrix} x \\ y \end{bmatrix}.$$

By considering the vector $z' = \begin{bmatrix} -x \\ -y \end{bmatrix}$ and $\lambda' = -\lambda$, then we have that

$$Az' = A \begin{bmatrix} -x \\ -y \end{bmatrix} = - \begin{bmatrix} 0 & B \\ B^T & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = -\lambda \begin{bmatrix} x \\ y \end{bmatrix} = \lambda' z'.$$

Exercise 12 [Converse for bipartite graphs]

(a) Argue that a graph with no odd length cycle is bipartite.

(b) Show that the statement “a graph has no odd length cycle” is equivalent to “for every odd k, $\mathrm{tr}(A^k) = 0$”.

(c) Deduce that if for every eigenvalue λ of A, there is another eigenvalue $-\lambda$, then the graph is bipartite.

(*Answer*)

(a)

(b)

(c) Consider any odd power k, then

$$\mathrm{tr}(A^k) = \sum_{i=1}^{n} \lambda_i^k = 0.$$

Hence, by Exercise 10 (c), we have that the graph has no cycle of odd length and so it is bipartite by (a).

Exercise 13 On Lecture 11/slide 10 (Example 1) we determined the spectrum of the adjacency matrix A for the complete graph (a.k.a. clique) of size 3. Here we would like to generalise this to any complete graph of size $n > 3$. Prove that the spectrum consists of eigenvalues $n-1$ (with multiplicity 1) and -1 (with multiplicity $n-1$).

Exercise 14 [Perron-Frobenius] Let G by a connected graph with adjacency matrix A with eigenvectors $x_1, \ldots, x_n$ and eigenvalues $\lambda_1, \ldots, \lambda_n$, then show that

(a) $\lambda_1 \geq -\lambda_n$,

(b) $\lambda_1 > \lambda_2$,

(c) There exists an eigenvector x_1 which has all its entries > 0.

2.2 Laplacian matrix

Exercise 15 [Factorisation] Consider the unormalised Laplacian matrix $L = D - A$ and the incident matrix $M \in \mathbb{R}^{n \times m}$ defined as $M_{ue} = \mathbf{1}_{u \in e}$ (i.e., indicates which edges contain which vertices). Show that

$$L = M^T M.$$

Exercise 16 Consider an undirected, d-regular graph G and the matrices A_G and L_G.

(a) Show that the two matrices have the same eigenvectors.

(b) Describe the correspondence between their eigenvalues.

(*Answer*) Assume that x is an eigenvector and λ is its corresponding eigenvalue. Then,

$$Ax = \lambda x.$$

Then,

$$Lx = \left(I - \frac{1}{d}A\right)x = x - \frac{\lambda}{d}x = \left(1 - \frac{\lambda}{d}\right)x.$$

So x is still an eigenvector and $1 - \lambda/d$ is an eigenvalue.

Exercise 17 Consider a d-regular graph and its Laplacian matrix L.

(a) Using the quadratic form, show that for any vector $x \in \mathbb{R}^n$ (with $x \neq 0$),

$$\frac{x^T L x}{x^T x} \leq 2.$$

(b) Deduce that $\lambda_n \leq 2$.

(*Answer*)

(a) We will use the inequality

$$(\alpha - \beta)^2 \leq 2 \cdot \alpha^2 + 2 \cdot \beta^2,$$

which holds since

$$\alpha^2 - 2\alpha\beta + \beta^2 \leq 2 \cdot \alpha^2 + 2 \cdot \beta^2 \Leftrightarrow 0 \leq 2 \cdot \alpha^2 + 2\alpha\beta + 2 \cdot \beta^2 \Leftrightarrow 0 \leq (\alpha + \beta)^2.$$

For any vector $x \in \mathbb{R}^n$, we have that

$$x^T L x = \frac{1}{d} \sum_{\{u,v\} \in E} (x_u - x_v)^2 \leq \frac{2}{d} \cdot \sum_{\{u,v\} \in E} (x_u^2 + x_v^2) = \frac{2}{d} \cdot \sum_{u \in V} d \cdot x_u^2 = 2 \cdot x^T x.$$

Hence,

$$\frac{x^T L x}{x^T x} \leq 2.$$

(b) By the Courant-Fischer theorem, we conclude that $\lambda_n \leq 2$.

Exercise 18 Show that if G is an undirected, d-regular, connected and bipartite graph, then the largest eigenvalue λ_n of the Laplacian matrix satisfies $\lambda_n = 2$ (this proves one direction of the fourth statement in the Lemma from Lecture 11/slide 14).

(*Answer*) Let V_1 and V_2 be the two components of the bipartite graph. We define the vector $f \in \mathbb{R}^d$ with

$$f_i = \begin{cases} 1 & \text{if } i \in V_1, \\ -1 & \text{if } i \in V_2. \end{cases}$$

By Exercise 16, it suffices to show that f is eigenvector of A. We begin by noting that

$$(Af)_i = \sum_{j=1}^{n} A_{i,j} f_j = \sum_{j \in N(i)} f_j.$$

For any $i \in V_1$, we have that $N(i) = d$ and all neighbours $j \in V_2$, so

$$(Af)_i = -d = -d \cdot f_i.$$

For any $i \in V_2$, we have that $N(i) = d$ and all neighbours $j \in V_2$, so

$$(Af)_i = d = -d \cdot f_i.$$

Hence, we conclude that $Af = -df$, and so f is an eigenvector of A with eigenvalue $\lambda = -d$. By Exercise 16, it follows that f is an eigenvector of L with eigenvalue $\lambda' = 1 - \frac{-d}{d} = 2$.
By Exercise 17, it follows that $\lambda' = 2$ is also the *largest* eigenvalue of the Laplacian.

Exercise 19 Redo Exercise 18 without assuming that the graph is d-regular.

(*Answer*) Recall that

$$L = I - D^{-1/2} A D^{-1/2},$$

and in particular

$$L_{u,v} = \frac{1}{\sqrt{\deg(u) \cdot \deg(v)}}.$$

Next, define the vector

$$f_i = \begin{cases} \sqrt{\deg(i)} & \text{if } i \in V_1, \\ -\sqrt{\deg(i)} & \text{if } i \in V_2. \end{cases}$$

Using a similar calculation as in Exercise 18, you can show that $Lf = 2f$.

Exercise 20 Consider the transition matrix of a lazy random walks $\tilde{P} = (P + I)/2$ on a d-regular graph (here I is the $n \times n$ identity matrix and P is the transition matrix of a simple random walk).

(a) Using Exercise 16 and that the eigenvalues of L are in $[0, 2]$, argue that the eigenvalues of A are in $[-d, d]$.

(b) Prove that all eigenvalues of $\tilde{P}$ are non-negative.

3 Conductance

Exercise 21 [Conductance of graphs]

(a) Compute the conductance of the *complete graph* K_n.

(b) Compute the conductance of the *cycle* C_n.

(c) Compute the conductance of a path P_n.

(d) Compute the conductance of a 2D grid.

(e) (+) Compute the conductance of a 3D grid.

Exercise 22

(a) Prove that for every $n > 2$ there is an unweighted, undirected n-vertex graph with conductance 1.

(b) (+) Can you characterise all graphs with that property?

Exercise 23 Prove that for any d-regular graph with $n \to \infty$ being large, the conductance satisfies $\Phi(G) \leq \frac{1}{2} + o(1)$.
Hint: Use the probabilistic method to construct a set S with the required conductance. First obtain bounds for $||S| - n/2|$ and then for $|E(S, S^c) - |E|/2|$.

(*Answer*) It suffices to construct a set S with small conductance. We construct the set S by adding each vertex u to S independently with probability $1/2$. In order to upper bound its conductance, we will try to bound both terms in the fraction

$$\Phi(S) = \frac{|E(S, S^c)|}{\min\{|S|, |S^c|\} \cdot d}.$$

We begin by lower bounding $\min\{|S|, |S^c|\}$. Note that $\mathbf{E}\left[\,|S|\,\right] = \frac{n}{2}$ and so by the nice Chernoff bound, we have that

$$\mathbf{Pr}\left[\left||S| - \frac{n}{2}\right| \leq t\right] \geq 2 \cdot e^{-2t^2/n}.$$

By choosing $t := \sqrt{n \log n}$, we have that

$$\mathbf{Pr}\left[\left||S| - \frac{n}{2}\right| \leq \sqrt{n \log n}\right] \geq 2 \cdot e^{-2 \log n} = \frac{2}{n^2}.$$

When $||S| - \frac{n}{2}| \leq \sqrt{n \log n}$, it also holds that

$$\left||S^c| - \frac{n}{2}\right| = \left|n - |S| - \frac{n}{2}\right| = \left||S| - \frac{n}{2}\right| \leq \sqrt{n \log n}.$$

Hence, w.h.p. it holds that

$$\min\{|S|, |S^c|\} \cdot d \geq \left(\frac{n}{2} - \sqrt{n \log n}\right) \cdot d. \tag{2}$$

For the numerator, we have that $E(S, S^c) = \frac{nd}{4}$. Now, we cannot use the nice Chernoff bound as there is dependence between the edges, namely if we remove u from S then up to d edges may be added or removed from $E(S, S^c)$. So, instead we apply McDiarmid's inequality with Lipschitz's constants $c_i := d$ and $t := d\sqrt{n \log n}$

$$\mathbf{Pr}\left[|E(S, S^c)| \geq \frac{nd}{4} + d\sqrt{n \log n}\right] \leq \exp\left(-2 \cdot \frac{d^2 n \log n}{n \cdot d^2}\right) = \frac{1}{n^2}. \tag{3}$$

Hence, taking the union bound between 2 and 3, we have that with probability at least $1 - \frac{3}{n^2}$, it holds that

$$\Phi(S) = \frac{|E(S, S^c)|}{\min\{|S|, |S^c|\} \cdot d} \leq \frac{\frac{nd}{4} + d\sqrt{n \log n}}{\left(\frac{n}{2} - \sqrt{n \log n}\right) \cdot d} \leq \frac{1/4 + \sqrt{\frac{\log n}{n}}}{1/2 - \sqrt{\frac{\log n}{n}}} \leq \frac{1}{2} - o(1).$$

Exercise 24 Consider an undirected, and d-regular graph $G = (V, E)$ with conductance $\Phi > 0$. In this exercise, you will show that the diameter of the graph is at most $\mathcal{O}(\frac{\log n}{\Phi})$.

(a) Consider an arbitrary vertex $u \in V$ and let $S_0 := \{u\}$ and $S_i := B_{\leq i}(u)$, the set of nodes at a distance of at most i from u. Show that for any S_i with $|S_i| \leq n/2$, it holds that

$$|E(S_i, S_i^c)| \geq \Phi \cdot |S_i| \cdot d,$$

and that

$$|S_{i+1}| \geq |S_i| \cdot (1 + \Phi).$$

(b) Using that $\log(1+z) \geq (1/2) \cdot z$ (for any $z \in [0,1]$), deduce that $|S_i| > n/2$ for $i > 2 \cdot \frac{\log n}{\Phi}$.

(c) Deduce that there is no pair of vertices at a distance $> \frac{4 \log n}{\Phi}$.

(*Answer*)

(a) By the definition of conductance, we have that since $|S_i| \leq n/2$,

$$\Phi(S_i) \geq \Phi \Rightarrow \frac{|E(S_i, S_i^c)|}{d \cdot |S_i|} \geq \Phi \Rightarrow |E(S_i, S_i^c)| \geq \Phi \cdot |S_i| \cdot d. \tag{4}$$

Further, we have that

$$|S_{i+1}| = |S_{i+1} \setminus S_i| + |S_i|. \tag{5}$$

Note that because of the distance property every node $u \in S_{i+1} \setminus S_i$ is connected to some node $v \in S_i$, i.e., $\{u, v\} \in E(S_i, S_i^c)$. Using that the graph is d-regular, (so u could be reachable by at most d nodes) it holds that

$$|S_{i+1} \setminus S_i| \geq \frac{1}{d} \cdot |E(S_i, S_i^c)|.$$

Combining with inequalities 4 and 5, we have that

$$|S_{i+1}| = \frac{1}{d} \cdot |E(S_i, S_i^c)| + |S_i| \geq \Phi \cdot |S_i| + |S_i| = |S_i| \cdot (1 + \Phi).$$

(b) As long as $|S_i| \leq n/2$, we can apply the previous part and obtain that

$$|S_{i+1}| \geq |S_i| \cdot (1+\Phi) \geq |S_{i-1}| \cdot (1+\Phi)^2 \geq \ldots \geq |S_0| \cdot (1+\Phi)^{i+1}.$$

So for any $i > 2\frac{\log n}{\Phi}$, it either holds that $|S_i| > n/2$ or

$$|S_i| \geq (1+\Phi)^i = \exp\left(i \cdot \log(1+\Phi)\right) \geq \exp\left(i \cdot \frac{\Phi}{2}\right) > \exp\left(2\frac{\log n}{\Phi} \cdot \frac{\Phi}{2}\right) = n,$$

which cannot hold. Hence, $|S_i| > n/2$.

(c) Consider any pair of two vertices u and v. Then by the previous part, we have that starting from vertex u in $2 \cdot \frac{\log n}{\Phi}$ steps, we will reach $> n/2$ of the vertices (call this set S_u). Similarly, starting from vertex v in $2 \cdot \frac{\log n}{\Phi}$ steps, we will reach $> n/2$ of the vertices (call this set S_v).

By the pigeonhole principle, it holds that $S_u \cap S_v \neq \emptyset$ and so there is one vertex k reachable from both u and v in $2 \cdot \frac{\log n}{\Phi}$ steps. Since the graph is undirected, we can combine the path from u to k and from k to v to construct a path from u to v with length at most $4 \cdot \frac{\log n}{\Phi}$.