Randomised Algorithms Supervision 4
Spectral graph theory: Solution Notes

1 Linear Algebra

**Exercise 1 [Properties of dagger]** Prove the following properties of the conjugate transpose, defined for any \( x \in \mathbb{C}^n \) as \( x^\dagger = (x_1^*, \ldots, x_n^*) \).

(a) For any two vectors \( x, y \in \mathbb{C}^n \), it holds that \( x^\dagger y = (y^\dagger x)^\dagger \).

(b) For any vector \( x \in \mathbb{C}^n \), it holds that \( x^\dagger x \geq 0 \).

(*Answer*)

(a) By writing out the definition of the two vectors, we have that
\[
(y^\dagger x)^\dagger = (y^\dagger x)^* = \left( \sum_{i=1}^{n} y_i^* x_i \right)^* = \sum_{i=1}^{n} (y_i^* x_i)^* = \sum_{i=1}^{n} y_i^* x_i = x^\dagger y.
\]

(b) For any vector \( x \in \mathbb{C}^n \),
\[
x^\dagger x = \sum_{i=1}^{n} x_i^* x_i = \sum_{i=1}^{n} |x_i|^2 \geq 0.
\]

**Extended Note 1** A matrix \( A \in \mathbb{C}^{n \times n} \) is **Hermitian** if \( A^\dagger = A \). The notation \( A^\dagger \) means the complex conjugate of \( A \), i.e., \( A^\dagger = (A^T)^* \). If you prefer you can attempt the following exercises, by assuming that \( A \) is a matrix with real entries and then \( A^\dagger = A^T = A \), meaning that the matrix is **symmetric**.

**Exercise 2 [Real eigenvalues]** Consider a Hermitian matrix \( A \in \mathbb{R}^{n \times n} \), then all its eigenvalues are real.

(*Answer*) Let \( A \) be a Hermitian matrix. Then, for any vector \( x \in \mathbb{C}^n \), it holds that
\[
(Ax)^\dagger x = x^\dagger A^\dagger x = x^\dagger (Ax). \tag{1}
\]
Assuming that \( x \) is an eigenvector corresponding to eigenvalue \( \lambda \), i.e., \( Ax = \lambda x \). Then, we have that
\[
(Ax)^\dagger x = (\lambda x)^\dagger x = \lambda^* x^\dagger x
\]
and
\[
x^\dagger (Ax) = x^\dagger \lambda x = \lambda x^\dagger x
\]
By (1) we have that
\[
\lambda x^\dagger x = \lambda^* x^\dagger x \Rightarrow (\lambda - \lambda^*) x^\dagger x = 0.
\]
Hence, since \( x^\dagger x > 0 \) (as \( x \neq 0 \)), we have that \( \lambda = \lambda^* \) and hence \( \lambda \) is real.

**Exercise 3 [Orthogonal eigenvectors]** Consider a Hermitian matrix \( A \in \mathbb{C}^{n \times n} \) and let \( x, y \in \mathbb{C}^n \) be two eigenvectors corresponding to different eigenvalues \( \lambda \neq \lambda' \). Then, \( x \perp y \).

(*Answer*) Since \( A \) is Hermitian, we have that
\[
(Ax)^\dagger y = x^\dagger A^\dagger y = x^\dagger Ay = x^\dagger (Ay).
\]
Exercise 4 [Spectral theorem] Let $A \in \mathbb{R}^{n \times n}$ be a symmetric matrix.

(a) Let $k \leq n-1$ and let $x_1, \ldots, x_k$ be orthogonal eigenvectors of $A$. Then, there exists an eigenvector $x_{k+1}$ that is orthogonal to $x_1, \ldots, x_k$.

(b) Prove the spectral theorem.

(c) Argue that $A$ can be written as $XDX^{-1}$ for some matrix $X$ and a diagonal matrix $D$.

Exercise 5 [Inverse matrix] Let $A \in \mathbb{R}^{n \times n}$ be a symmetric matrix with (orthonormal) eigenvectors $x_1, \ldots, x_n$ corresponding to eigenvalues $\lambda_1, \ldots, \lambda_n$. Prove that:

(a) Let $x \in \mathbb{R}^n$ be arbitrary. By writing $x = (x^T x_1) x_1 + \cdots + (x^T x_n) x_n$, show that $x_1 x_1^T + \cdots + x_n x_n^T = I$.

(b) By writing $Ax = AIx$, show that $A = \lambda_1 x_1 x_1^T + \cdots + \lambda_n x_n x_n^T$.

(c) Show that if $\lambda_1 \neq 0, \ldots, \lambda_n \neq 0$, then $A^{-1} = \frac{1}{\lambda_1} x_1 x_1^T + \cdots + \frac{1}{\lambda_n} x_n x_n^T$.

Exercise 6 [Power of a matrix] Consider a real symmetric matrix $A \in \mathbb{R}^{n \times n}$ with eigenvectors $x_1, \ldots, x_n$ and eigenvalues $\lambda_1, \ldots, \lambda_n$. Then, for any $k \in \mathbb{N}_+ = \{1, 2, \ldots\}$, the eigenvectors of $A^k$ are $x_1, \ldots, x_n$ and the eigenvalues $\lambda_1^k, \ldots, \lambda_n^k$.

Exercise 7 [Trace of a matrix] The trace of a matrix is defined as $\text{tr}(A) = \sum_{i=1}^{n} A_{ii}$.

(a) Show that for any two matrices $X$ and $Y$, we have that $\text{tr}(XY) = \text{tr}(YX)$.

(b) Show that for any real symmetric matrix $A$, $\text{tr}(A) = \sum_{i=1}^{n} \lambda_i$.

(Answer)

(a) Using the definition of the trace, we have that $\text{tr}(XY) = \sum_{i=1}^{n} (XY)_{ii} = \sum_{i=1}^{n} \sum_{j=1}^{n} X_{ij} Y_{ji} = \sum_{j=1}^{n} \sum_{i=1}^{n} X_{ij} Y_{ji} = \sum_{j=1}^{n} \sum_{i=1}^{n} Y_{ji} X_{ij} = \text{tr}(YX)$.

(b) By writing $A = XDX^{-1}$, we have that $\text{tr}(XDX^{-1}) = \text{tr}(XX^{-1}D) = \text{tr}(ID) = \text{tr}(D) = \sum_{i=1}^{n} \lambda_i$.

Exercise 8 [Determinant of symmetric matrix] Consider a real symmetric matrix $A$. Show that $\det(A) = \prod_{i=1}^{n} \lambda_i$.

Hint: Use the property that $\det(AB) = \det(A) \cdot \det(B)$. 2
By writing $A = XD X^{-1}$ we have that
\[
\det(A) = \det(XD X^{-1}) = \det(X)\det(D)\det(X^{-1}) = \det(D) = \prod_{i=1}^{n} \lambda_i,
\]
using that $\det(X^{-1}) = (\det(X))^{-1}$ and that the determinant of a diagonal matrix is just the product of the entries of the diagonal.

## 2 Graph matrices

### 2.1 Adjacency matrix

**Exercise 9 [Basic properties]** Consider the adjacency matrix of an undirected graph.

(a) Show that $\text{deg}(v_i) = \sum_{j=1}^{n} A_{ij}$.

(b) Show that the adjacency matrix of a $d$-regular graph has eigenvalue $d$.

**Exercise 10 [Counting paths]** Consider the adjacency matrix $A$ of a graph $G$.

(a) Show that if $(A^k)_{ij} > 0$ for some integer $k > 0$ then there is a path of length $k$ connecting $i$ and $j$.

(b) Show that $(A^k)_{ij}$ also gives the number of paths connecting $i$ and $j$ with $k$ hops.

(c) Interpret $\text{tr}(A^k) > 0$.

**Exercise 11 [Bipartite graphs]** Show that for any bipartite graph $G$ with adjacency matrix $A$, if $\lambda > 0$ is an eigenvalue then $-\lambda$ is also an eigenvalue.

**Answer** Let $G$ be a bipartite graph where the left part has $k$ vertices and the right part has $n-k$. We re-index the $n$ nodes of the graph such that the vertices of the left part are $1, \ldots, k$ and the vertices of the right part are $k+1, \ldots, n$. Then, the adjacency matrix can be written as
\[
A = \begin{bmatrix} 0 & B \\ B^T & 0 \end{bmatrix}.
\]

Let $\begin{bmatrix} x \\ y \end{bmatrix}$ be the eigenvector of $A$ with eigenvalue $\lambda$, then
\[
A \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 & B \\ B^T & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} Bx \\ B^Ty \end{bmatrix} = \lambda \begin{bmatrix} x \\ y \end{bmatrix}.
\]

By considering the vector $z' = \begin{bmatrix} -x \\ -y \end{bmatrix}$ and $\lambda' = -\lambda$, then we have that
\[
A z' = A \begin{bmatrix} -x \\ -y \end{bmatrix} = - \begin{bmatrix} 0 & B \\ B^T & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = -\lambda \begin{bmatrix} x \\ y \end{bmatrix} = \lambda' z'.
\]

**Exercise 12 [Converse for bipartite graphs]**

(a) Argue that a graph with no odd length cycle is bipartite.

(b) Show that the statement “a graph has no odd length cycle” is equivalent to “for every odd $k$, $\text{tr}(A^k) = 0$”.

(c) Deduce that if for every eigenvalue $\lambda$ of $A$, there is another eigenvalue $-\lambda$, then the graph is bipartite.

**Answer**
(a) 
(b) 

(c) Consider any odd power \( k \), then

\[
\text{tr}(A^k) = \sum_{i=1}^{n} \lambda_i^k = 0.
\]

Hence, by Exercise 10(c) we have that the graph has no cycle of odd length and so it is bipartite by (a).

**Exercise 13** On Lecture 11/slide 10 (Example 1) we determined the spectrum of the adjacency matrix \( A \) for the complete graph (a.k.a. clique) of size 3. Here we would like to generalise this to any complete graph of size \( n > 3 \). Prove that the spectrum consists of eigenvalues \( n-1 \) (with multiplicity 1) and \(-1\) (with multiplicity \( n-1 \)).

**Exercise 14** [Perron-Frobenius] Let \( G \) be a connected graph with adjacency matrix \( A \) with eigenvectors \( x_1, \ldots, x_n \) and eigenvalues \( \lambda_1, \ldots, \lambda_n \), then show that

(a) \( \lambda_1 \geq -\lambda_n \),
(b) \( \lambda_1 > \lambda_2 \),
(c) There exists an eigenvector \( x_1 \) which has all its entries > 0.

2.2 Laplacian matrix

**Exercise 15** [Factorisation] Consider the unnormalised Laplacian matrix \( L = D - A \) and the incident matrix \( M \in \mathbb{R}^{n \times m} \) defined as \( M_{ue} = 1_{u \in e} \) (i.e., indicates which edges contain which vertices). Show that

\[
L = M^T M.
\]

**Exercise 16** Consider an undirected, \( d \)-regular graph \( G \) and the matrices \( A_G \) and \( L_G \).
(a) Show that the two matrices have the same eigenvectors.
(b) Describe the correspondence between their eigenvalues.

(Answer) Assume that \( x \) is an eigenvector and \( \lambda \) is its corresponding eigenvalue. Then,

\[
Ax = \lambda x.
\]

Then,

\[
Lx = \left( I - \frac{1}{d} A \right) x = x - \frac{\lambda}{d} x = \left( 1 - \frac{\lambda}{d} \right) x.
\]

So \( x \) is still an eigenvector and \( 1 - \lambda/d \) is an eigenvalue.

**Exercise 17** Consider a \( d \)-regular graph and its Laplacian matrix \( L \).
(a) Using the quadratic form, show that for any vector \( x \in \mathbb{R}^n \) (with \( x \neq 0 \)),

\[
\frac{x^T Lx}{x^T x} \leq 2.
\]

(b) Deduce that \( \lambda_n \leq 2 \).
Prove that for any \( d \)-regular graph, the largest eigenvalue of the Laplacian \( L \) satisfies \( \lambda_n \leq 2 \).

**Exercise 18** Show that if \( G \) is an undirected, \( d \)-regular, connected and bipartite graph, then the largest eigenvalue \( \lambda_n \) of the Laplacian matrix satisfies \( \lambda_n = 2 \) (this proves one direction of the fourth statement in the Lemma from Lecture 11/slide 14).

**Exercise 19** Redo Exercise 18 without assuming that the graph is not \( d \)-regular.
Exercise 20 Consider the transition matrix of a lazy random walks $\tilde{P} = (P + I)/2$ on a $d$-regular graph (here $I$ is the $n \times n$ identity matrix and $P$ is the transition matrix of a simple random walk).

(a) Using Exercise 16 and that the eigenvalues of $L$ are in $[0, 2]$, argue that the eigenvalues of $A$ are in $[-d, d]$.

(b) Prove that all eigenvalues of $\tilde{P}$ are non-negative.

3 Conductance

Exercise 21 [Conductance of graphs]

(a) Compute the conductance of the complete graph $K_n$.

(b) Compute the conductance of the cycle $C_n$.

(c) Compute the conductance of a path $P_n$.

(d) Compute the conductance of a 2D grid.

(e) (+) Compute the conductance of a 3D grid.

Exercise 22

(a) Prove that for every $n > 2$ there is an unweighted, undirected $n$-vertex graph with conductance 1.

(b) (+) Can you characterise all graphs with that property?

Exercise 23 Prove that for any $d$-regular graph with $n \to \infty$ being large, the conductance satisfies

$$\Phi(G) \leq \frac{1}{2} + o(1).$$

Hint: Use the probabilistic method to construct a set $S$ with the required conductance. First obtain bounds for $||S| - n/2|$ and then for $|E(S, S^c) - |E||/2|$.

Exercise 24 Consider an undirected, and $d$-regular graph $G = (V, E)$ with conductance $\Phi > 0$. In this exercise, you will show that the diameter of the graph is at most $O(\frac{\log n}{\Phi})$.

(a) Consider an arbitrary vertex $u \in V$ and let $S_0 := \{u\}$ and $S_i := B_{\leq i}(u)$, the set of nodes at a distance of at most $i$ from $u$. Show that for any $S_i$ with $|S_i| \leq n/2$, it holds that

$$|E(S_i, S^c_i)| \geq \Phi \cdot |S_i| \cdot d,$$

and that

$$|S_{i+1}| \geq |S_i| \cdot (1 + \Phi).$$

(b) Using that $\log(1 + z) \geq (1/2) \cdot z$ (for any $z \in [0, 1]$), deduce that $|S_i| > n/2$ for $i > 2 \cdot \frac{\log n}{\Phi}$.

(c) Deduce that there is no pair of vertices at a distance $> \frac{4 \log n}{\Phi}$.