## 1 LPs and Simplex

**Exercise 1** Convert the following LP into slack form. Also state the set of basic and non-basic variables.

\[
\begin{align*}
\text{maximise} & \quad 2x_1 - 6x_3 \\
\text{subject to} & \quad x_1 + x_2 - x_3 \leq 7 \\
& \quad 3x_1 - x_2 \geq 8 \\
& \quad -x_1 + 2x_2 + 2x_3 \geq 0 \\
& \quad x_1, x_2, x_3 \geq 0
\end{align*}
\]

[Source: CLRS: 29.1-5]

**Exercise 2** Show that the following LP is infeasible:

\[
\begin{align*}
\text{maximise} & \quad 3x_1 - 2x_2 \\
\text{subject to} & \quad x_1 + x_2 \leq 2 \\
& \quad -2x_1 - 2x_2 \leq -10 \\
& \quad x_1, x_2 \geq 0
\end{align*}
\]

[Source: CLRS: 29.1-6]

**Exercise 3** Show that the following LP is unbounded:

\[
\begin{align*}
\text{maximise} & \quad x_1 - x_2 \\
\text{subject to} & \quad -2x_1 + x_2 \leq -1 \\
& \quad -x_1 - 2x_2 \leq -2 \\
& \quad x_1, x_2 \geq 0
\end{align*}
\]

[Source: CLRS: 29.1-7]

**Exercise 4** Find a linear program which has more than one optimal solution.

**Exercise 5** Solve the following linear program using **SIMPLEX**:

\[
\begin{align*}
\text{maximise} & \quad 5x_1 - 3x_2 \\
\text{subject to} & \quad x_1 - x_2 \leq 1 \\
& \quad 2x_1 + x_2 \leq 2 \\
& \quad x_1, x_2 \geq 0
\end{align*}
\]

[Source: CLRS: 29.3-6]
Exercise 6 Solve the following linear program using SIMPLEX:

\[
\text{maximise} \quad x_1 + 3x_2 \\
\text{subject to} \quad x_1 - x_2 \leq 1 \\
2x_1 + x_2 \leq 2 \\
x_1, x_2 \geq 0
\]

[Source: CLRS: 29.3-6]

Exercise 7 Attempt 2015P7Q2(a)-(b).

Exercise 8 Attempt 2018P27Q1(a)-(b).

Exercise 9 [Duality] Attempt 2018P27Q1(c).


Exercise 11 Attempt 2020P8Q1(a).

2 Formulating problems as Linear Programs

Exercise 12 [Shortest path] Consider the linear program for the shortest path problem from s to t.

1. What happens if there is a negative-weight cycle?

2. Prove that, if there are no negative-weight cycles, the optimal solution \( d_t \) of the linear program equals the correct distance \( d_t \).

3. How would your formulate the single-source shortest path problem as a linear program?

Exercise 13 [Multi-commodity flow] In the multi-commodity flow problem, there is a graph \( G = (V,E) \) where edges have capacities \( c(u,v) \). There are \( k \) commodities \( K_1, \ldots, K_k \), where \( K_i = (s_i,t_i,d_i) \), meaning that there is a demand of \( d_i \) from source \( s_i \) to destination \( t_i \). Formulate this as a linear program (LP).

Exercise 14 [Minimum Spanning Tree]

(a) Formulate the (undirected) Minimum Spanning Tree problem as an integer program (possibly using an exponential number of constraints).

Hint: Think about subtour eliminations.

(b) Formulate the linear relaxation of the integer program.

(c) Show that given a fractional solution to the linear relaxation, there is a polynomial time algorithm to convert the fractional solution into an integer one with the same cost. What does this show?

(Answer)

(a) Let \( x_{uv} \in \{0,1\} \) indicate whether the edge \((u,v)\) is in the output MST. Then the cost of the MST is \( \sum_{u,v \in E} x_{uv} w_{uv} \). We need to add constraints so that the indicators do not form a cycle. This means that every set of vertices \( S \subseteq V \) must have at most \( |S| - 1 \) internal edges:

\[
\sum_{(u,v) \in E(S,S)} x_{uv} \leq |S| - 1.
\]
Hence, putting it all together, we have that:

\[
\begin{aligned}
\text{maximise} & \quad \sum_{uv \in E} x_{uv}w_{uv} \\
\text{subject to} & \quad x_{uv} \in \{0, 1\} \quad \forall uv \in E \\
& \quad \sum_{uv \in E} x_{uv} = |V| - 1 \\
& \quad \sum_{(u,v) \in E(S,S)} x_{uv} \leq |S| - 1 \quad \forall \emptyset \subset S \subseteq V
\end{aligned}
\]

(b) If all values \(x_{uv}\) are integer then it means that the induced solution is indeed a spanning tree and it indeed has the minimum value.

Otherwise, a cycle is formed and this cycle must have one edge edge \(e\) with \(x_e < 1\). By picking the heaviest edge \(e'\) on the cycle and moving as much of \(x_{e'}\) to \(x_e\) making one of the two either 0 or 1, we obtain a solution with one integer solution value and the same (or smaller) cost.

\textit{Note:} If the edge weights are unique, then the MST is unique and we can argue about integrality since there can be no cycle.

\textbf{Exercise 15 [Min-Cost Bipartite Matching]} In the \textit{min-cost bipartite matching} problem, we are given a bipartite graph \(G = (X \cup Y, E)\) with \(|X| = |Y|\) and a weight function \(w : X \times Y \to \mathbb{R}\) and our goal is to find the bipartite matching with the minimum cost.

(a) Formulate this as an integer program.
(b) Formulate the linear relaxation of the integer program.
(c) (+) Give a polynomial-time algorithm for converting a fractional solution of the linear relaxation to a solution to the original problem.

\textit{Hint:} Given a fractional solution, modify the solution so that the number of integral values (0 or 1) in the solution increases, and the objective remains the same.

\textbf{Exercise 16 [Bounded Degree MST]} Formulate as an integer program the bounded degree MST problem, where we want to find the MST where each vertex has at most \(d\) neighbours.

\textbf{Exercise 17} Attempt [2017P7Q1](a)-(b).

\textbf{Exercise 18} Attempt [2020P8Q1](c).

\textbf{Exercise 19} Prove that the LP formulation of the Set-Cover problem (Lecture 10, slide 4) is feasible if and only if the Set-Cover instance has a feasible solution.

3 Convex sets

\textbf{Extended Note 1 [Convex set]} Recall a set \(S\) is convex if for every \(x, y \in S, \lambda x + (1 - \lambda)y \in S\) for all \(\lambda \in [0, 1]\).

\textbf{Exercise 20}
(a) Show that the intersection of two convex sets \(C_1\) and \(C_2\) are convex.
(b) Show that the union of two convex sets need not be convex.

\textbf{Exercise 21}
(a) Prove that for any \(a_1, \ldots, a_n, b \in \mathbb{R}\), we have that the set
\[
\mathcal{A} := \{x \in \mathbb{R}^n : a_1x_1 + \ldots + a_nx_n \leq b\}
\]
is convex.
(b) Prove that the set of feasible solutions of a linear program forms a convex set.

(c) Prove that the set of solutions of a linear problem forms a convex set.

4 (non-randomised) approximation algorithms

Exercise 22 [Vertex Cover]
(a) Analyse the greedy algorithm for the unweighted Vertex cover problem that achieves an approximation ratio of 2 (Slide 14 of Lecture 9).

(b) Bonus-Question: What is the problem behind the “more natural” greedy approach where instead of both endpoints of an uncovered edge, we only include one of the two endpoints into our cover?

Exercise 23 [Minimum Cardinality Maximal Matching] Design a factor 2 approximation algorithm for the problem of finding a minimum cardinality maximal matching in an undirected graph. A matching is maximal if no other edge can be added to the matching.

Hint: Use the fact that any maximal matching is at least half the maximum matching.

Exercise 24 [Maximum acyclic graph] Given a directed graph \( G = (V, E) \), pick a maximum cardinality set of edges from \( E \) so that the resulting subgraph is acyclic. Design an approximation algorithm for this problem.

(Answer) Consider the following algorithm:

1. Arbitrarily number of the vertices of the graph.
2. Let \( F \) be the set of edges \((u, v)\) with \( u < v \) and \( B \) the set of edges \((u, v)\) with \( v < u \).
3. Return the largest of the two sets \( B \) and \( F \).

It is clear that both edge sets induce an acyclic graph. The two sets satisfy \(|B| + |F| = |E|\) and so one of the two must have size at least \(|E|/2\). Since the optimal answer may have at most \(|E|\) edges, this proves an approximation ratio of 2.

Exercise 25 [Next-Fit for Bin-Packing] Consider the Next-Fit heuristic for the bin-packing problem. Show that it is a 2-approximation algorithm.

Exercise 26 Attempt [2017P9Q1].

Exercise 27 [Metric TSP] Consider TSP problem on a graph \( G = (V, E) \) with cost function \( c : V \times V \rightarrow \mathbb{R} \), which satisfies \( c(u, v) + c(v, w) \geq c(u, w) \) for all \( u, v, w \in V \).

Consider the algorithm that picks an arbitrary root \( r \) and then finds an MST from that root, and as a solution returns a walk of the MST (e.g. the pre-order traversal). Show that this algorithm achieves a 2-approximation for this version of the TSP.

(Answer) See these old slides.

Exercise 28 Attempt [2015P7Q2](c). Hint: First look at Exercise 27

Exercise 29 [Christophides’ algorithm] (+) Read about Christophides’ algorithm for the TSP problem from these slides. Show that it achieves a 3/2-approximation ratio.
Exercise 30  Attempt [2018P9Q1].

Exercise 31  Consider an instance of the unweighted Set-Cover problem with the condition that no element \( x \in X \) appears in more than \( k \) many subsets. Design an approximation algorithm based on deterministic rounding which achieves an approximation ratio of at most \( \mathcal{O}(k) \).

5  Randomised approximation algorithms

Exercise 32  Consider a MAX-SAT formula where each clause has at least 4 literals. Design a randomised approximation algorithm and analyse its approximation ratio.

Exercise 33  [MAX-4-CNF] Attempt [2016P9Q1](b).

Exercise 34  Consider the randomised approximation algorithm for the weighted Set-Cover problem. Translate the algorithm from the course into one based on non-linear randomised rounding such that, given the LP solution \( y \), we directly round this LP solution to get a cover \( C \) which (i) covers all elements with probability \( 1 - 1/n \), and (ii) has an expected cost which is at most \( \mathcal{O}(\log n) \) times the cost of the optimal cover.

Hint: By non-linear we refer to the way of choosing the probability of setting a variable to 1. The randomised rounding rules in the lecture is linear in the sense that the probability is equal to the fractional value of the LP solution.

Exercise 35  Attempt [2017P7Q1](c).

Exercise 36  Attempt [2020P9Q1].

Exercise 37  Recall the randomised algorithm for Set-Cover presented in the lecture. As input, we have a Set-Cover instance with \( n \) elements; and let us additionally assume we have at most \( \text{poly}(n) \) many subsets (and also that we can cover all \( n \) elements, i.e., \( \bigcup_{S \in X} S = X \)). The algorithm achieves with probability \( 1/3 \) that the returned cover is correct and the cost of the cover is at most a factor of \( 4 \ln(n) \) away from the optimal cost (see Lecture 10, slide 9.2). Turn this into a randomised algorithm which satisfies the following three conditions:

1. The algorithm terminates in a time that is polynomial in the input size, with probability 1 (i.e., always).
2. The algorithm returns a correct solution, with probability 1 (i.e., always).
3. The expected approximation ratio is \( \mathcal{O}(\log n) \).

Exercise 38  Consider the analysis of the hybrid algorithm for MAX-Sat, specifically Lecture 10, slide 16, which was based on lower bounding \( \alpha_{\ell} \) and \( \beta_{\ell} \), as well as their average.

(a) Which algorithm would you use if the CNF-formula only contains clauses of length at least 3?

(b) Design an algorithm which works well for a CNF-formula that only contains clauses of length 1 or 3. Hint: Slightly modify the hybrid algorithm from the lecture.

Exercise 39  The following problem arises in telecommunication networks, and is known as the SONET ring loading problem. The network consists of a cycle on \( n \) nodes, numbered 0 through \( n - 1 \) clockwise around the cycle. Some set \( C \) of calls is given; each call is a pair \((i,j)\) originating at node \( i \) and destined to

5
node \( j \). The call can be routed either clockwise or counterclockwise around the ring. The objective is to route the calls so as to minimise the total load on the network. The load \( L_i \) on the link \((i, (i + 1) \mod n)\) is the number of calls routed through \((i, (i + 1) \mod n)\), and the total load is

\[
\max_{1 \leq i \leq n} L_i.
\]

Give a 2-approximation algorithm for the SONET ring loading problem.

[Source: Williamson, Shmoys, Exercise 4.1]

6 Derandomisation

**Exercise 40** Apply the derandomisation trick based on conditional expectation for the randomised 2-approximation algorithm for MAX-CUT. Can you interpret the resulting algorithm?

**Exercise 41** Apply the derandomisation approach based on conditional expectation to the MAX-CNF problem.