Randomised Algorithms Supervision 4
Spectral graph theory

1 Linear Algebra

Exercise 1 [Properties of dagger] Prove the following properties of the conjugate transpose, defined for any $x \in \mathbb{C}^n$ as

$$x^\dagger = (x_1^*, \ldots, x_n^*).$$

(a) For any two vectors $x, y \in \mathbb{C}^n$, it holds that $x^\dagger y = (y^\dagger x)^\dagger$.

(b) For any vector $x \in \mathbb{C}^n$, it holds that $x^\dagger x \geq 0$.

Extended Note 1 A matrix $A \in \mathbb{C}^{n\times n}$ is *Hermitian* if $A^\dagger = A$. The notation $A^\dagger$ means the complex conjugate of A, i.e., $A^\dagger = (A^T)^*$. If you prefer you can attempt the following exercises, by assuming that A is a matrix with real entries and then $A^\dagger = A^T = A$, meaning that the matrix is *symmetric*.

Exercise 2 [Real eigenvalues] Consider a Hermitian matrix $A \in \mathbb{R}^{n\times n}$, then all its eigenvalues are real.

Exercise 3 [Orthogonal eigenvectors] Consider a Hermitian matrix $A \in \mathbb{C}^{n\times n}$ and let $x, y \in \mathbb{C}^n$ be two eigenvectors corresponding to different eigenvalues $\lambda \neq \lambda'$. Then, $x \perp y$.

Exercise 4 [Spectral theorem] Let $A \in \mathbb{R}^{n\times n}$ be a symmetric matrix.

(a) Let $k \leq n-1$ and let $x_1, \ldots, x_k$ be orthogonal eigenvectors of A. Then, there exists an eigenvector x_{k+1} that is orthogonal to $x_1, \ldots, x_k$.

(b) Prove the spectral theorem,

(c) Argue that A can be written as XDX^{-1} for some matrix X and a diagonal matrix D.

Exercise 5 [Inverse matrix] Let $A \in \mathbb{R}^{n\times n}$ be a symmetric matrix with (orthonormal) eigenvectors $x_1, \ldots, x_n$ corresponding to eigenvalues $\lambda_1, \ldots, \lambda_n$. Prove that:

(a) Let $x \in \mathbb{R}^n$ be arbitrary. By writing $x = (x^T x_1)x_1 + \ldots + (x^T x_n)x_n$, show that

$$x_1 x_1^T + \ldots x_n x_n^T = I.$$

(b) By writing $Ax = AIx$, show that

$$A = \lambda_1 x_1 x_1^T + \ldots \lambda_n x_n x_n^T.$$

(c) Show that if $\lambda_1 \neq 0, \ldots, \lambda_n \neq 0$, then

$$A^{-1} = \frac{1}{\lambda_1} x_1 x_1^T + \ldots + \frac{1}{\lambda_n} x_n x_n^T.$$

Exercise 6 [Power of a matrix] Consider a real symmetric matrix $A \in \mathbb{R}^{n\times n}$ with eigenvectors $x_1, \ldots, x_n$ and eigenvalues $\lambda_1, \ldots, \lambda_n$. Then, for any $k \in \mathbb{N}_{\geq 1}$, the eigenvectors of A^k are $x_1, \ldots, x_n$ and the eigenvalues $\lambda_1^k, \ldots, \lambda_n^k$.

Exercise 7 [Trace of a matrix] The *trace of a matrix* is defined as

$$\mathrm{tr}(A) = \sum_{i=1}^{n} A_{ii}.$$

(a) Show that for any two matrices X and Y, we have that $\mathrm{tr}(XY) = \mathrm{tr}(YX)$.

(b) Show that for any real symmetric matrix A, $\mathrm{tr}(A) = \sum_{i=1}^{n} \lambda_i$.

Exercise 8 [Determinant of symmetric matrix] Consider a real symmetric matrix A. Show that

$$\det(A) = \prod_{i=1}^{n} \lambda_i.$$

Hint: Use the property that $\det(AB) = \det(A) \cdot \det(B)$.

2 Graph matrices

2.1 Adjacency matrix

Exercise 9 [Basic properties] Consider the adjacency matrix of an undirected graph.

(a) Show that $\deg(v_i) = \sum_{j=1}^{n} A_{ij}$.

(b) Show that the adjacency matrix of a d-regular graph has eigenvalue d.

Exercise 10 [Counting paths] Consider the adjacency matrix A of a graph G.

(a) Show that if $(A^k)_{ij} > 0$ for some integer $k > 0$ then there is a path of length k connecting i and j.

(b) Show that $(A^k)_{ij}$ also gives the number of paths connecting i and j with k hops.

(c) Interpret $\mathrm{tr}(A^k) > 0$.

Exercise 11 [Bipartite graphs] Show that for any bipartite graph G with adjacency matrix A, if $\lambda > 0$ is an eigenvalue then $-\lambda$ is also an eigenvalue.

Exercise 12 [Converse for bipartite graphs]

(a) Argue that a graph with no odd length cycle is bipartite.

(b) Show that the statement "a graph has no odd length cycle" is equivalent to "for every odd k, $\mathrm{tr}(A^k) = 0$".

(c) Deduce that if for every eigenvalue λ of A, there is another eigenvalue $-\lambda$, then the graph is bipartite.

Exercise 13 On Lecture 11/slide 10 (Example 1) we determined the spectrum of the adjacency matrix A for the complete graph (a.k.a. clique) of size 3. Here we would like to generalise this to any complete graph of size $n > 3$. Prove that the spectrum consists of eigenvalues $n-1$ (with multiplicity 1) and -1 (with multiplicity $n-1$).

Exercise 14 [Perron-Frobenius] Let G by a connected graph with adjacency matrix A with eigenvectors $x_1, \dots, x_n$ and eigenvalues $\lambda_1, \dots, \lambda_n$, then show that

(a) $\lambda_1 \geq -\lambda_n$,

(b) $\lambda_1 > \lambda_2$,

(c) There exists an eigenvector x_1 which has all its entries > 0.

2.2 Laplacian matrix

Exercise 15 [Factorisation] Consider the unormalised Laplacian matrix $L = D - A$ and the incident matrix $M \in \mathbb{R}^{n\times m}$ defined as $M_{ue} = \mathbf{1}_{u\in e}$ (i.e., indicates which edges contain which vertices). Show that

$$L = M^T M.$$

Exercise 16 Consider an undirected, d-regular graph G and the matrices A_G and L_G.

(a) Show that the two matrices have the same eigenvectors.

(b) Describe the correspondence between their eigenvalues.

Exercise 17 Consider a d-regular graph and its Laplacian matrix L.

(a) Using the quadratic form, show that for any vector $x \in \mathbb{R}^n$ (with $x \neq 0$),

$$\frac{x^T Lx}{x^T x} \leq 2.$$

(b) Deduce that $\lambda_n \leq 2$.

Prove that for any d-regular graph, the largest eigenvalue of the Laplacian L satisfies $\lambda_n \leq 2$.

Exercise 18 Show that if G is an undirected, d-regular, connected and bipartite graph, then the largest eigenvalue λ_n of the Laplacian matrix satisfies $\lambda_n = 2$ (this proves one direction of the fourth statement in the Lemma from Lecture 11/slide 14).

Exercise 19 Redo Exercise 19 without assuming that the graph is not d-regular.

Exercise 20 Consider the transition matrix of a lazy random walks $\tilde{P} = (P + I)/2$ on a d-regular graph (here I is the $n \times n$ identity matrix and P is the transition matrix of a simple random walk).

(a) Using Exercise 17 and that the eigenvalues of L are in $[0, 2]$, argue that the eigenvalues of A are in $[-d, d]$.

(b) Prove that all eigenvalues of $\tilde{P}$ are non-negative.

3 Conductance

Exercise 21 [Conductance of graphs]

(a) Compute the conductance of the *complete graph* K_n.

(b) Compute the conductance of the *cycle* C_n.

(c) Compute the conductance of a path P_n.

(d) Compute the conductance of a 2D grid.

(e) (+) Compute the conductance of a 3D grid.

Exercise 22

(a) Prove that for every $n > 2$ there is an unweighted, undirected n-vertex graph with conductance 1.

(b) (+) Can you characterise all graphs with that property?

Exercise 23 Prove that for any d-regular graph with $n \to \infty$ being large, the conductance satisfies $\Phi(G) \leq \frac{1}{2} + o(1)$.
Hint: Use the probabilistic method to construct a set S with the required conductance. First obtain bounds for $||S| - n/2|$ and then for $|E(S, S^c) - |E|/2|$.

Exercise 24 Consider an undirected, and d-regular graph $G = (V, E)$ with conductance $\Phi > 0$. In this exercise, you will show that the diameter of the graph is at most $\mathcal{O}(\frac{\log n}{\Phi})$.

(a) Consider an arbitrary vertex $u \in V$ and let $S_0 := \{u\}$ and $S_i := B_{\leq i}(u)$, the set of nodes at a distance of at most i from u. Show that for any S_i with $|S_i| \leq n/2$, it holds that

$$|E(S_i, S_i^c)| \geq \Phi \cdot |S_i| \cdot d,$$

and that

$$|S_{i+1}| \geq |S_i| \cdot (1 + \Phi).$$

(b) Using that $\log(1 + z) \geq (1/2) \cdot z$ (for any $z \in [0, 1]$), deduce that $|S_i| > n/2$ for $i > 2 \cdot \frac{\log n}{\Phi}$.

(c) Deduce that there is no pair of vertices at a distance $> \frac{4 \log n}{\Phi}$.