1 Linear Algebra

**Exercise 1 [Properties of dagger]** Prove the following properties of the conjugate transpose, defined for any \( x \in \mathbb{C}^n \) as
\[
x^\dagger = (x_1^*, \ldots, x_n^*).
\]
(a) For any two vectors \( x, y \in \mathbb{C}^n \), it holds that \( x^\dagger y = (y^\dagger x)^\dagger \).
(b) For any vector \( x \in \mathbb{C}^n \), it holds that \( x^\dagger x \geq 0 \).

**Extended Note 1** A matrix \( A \in \mathbb{C}^{n \times n} \) is *Hermitian* if \( A^\dagger = A \). The notation \( A^\dagger \) means the complex conjugate of \( A \), i.e., \( A^\dagger = (A^T)^* \). If you prefer you can attempt the following exercises, by assuming that \( A \) is a matrix with real entries and then \( A^\dagger = A^T = A \), meaning that the matrix is *symmetric*.

**Exercise 2 [Real eigenvalues]** Consider a Hermitian matrix \( A \in \mathbb{R}^{n \times n} \), then all its eigenvalues are real.

**Exercise 3 [Orthogonal eigenvectors]** Consider a Hermitian matrix \( A \in \mathbb{C}^{n \times n} \) and let \( x, y \in \mathbb{C}^n \) be two eigenvectors corresponding to different eigenvalues \( \lambda \neq \lambda' \). Then, \( x \perp y \).

**Exercise 4 [Spectral theorem]** Let \( A \in \mathbb{R}^{n \times n} \) be a symmetric matrix.
(a) Let \( k \leq n - 1 \) and let \( x_1, \ldots, x_k \) be orthogonal eigenvectors of \( A \). Then, there exists an eigenvector \( x_{k+1} \) that is orthogonal to \( x_1, \ldots, x_k \).
(b) Prove the spectral theorem,
(c) Argue that \( A \) can be written as \( XDX^{-1} \) for some matrix \( X \) and a diagonal matrix \( D \).

**Exercise 5 [Inverse matrix]** Let \( A \in \mathbb{R}^{n \times n} \) be a symmetric matrix with (orthonormal) eigenvectors \( x_1, \ldots, x_n \) corresponding to eigenvalues \( \lambda_1, \ldots, \lambda_n \). Prove that:
(a) Let \( x \in \mathbb{R}^n \) be arbitrary. By writing \( x = (x^T x_1)x_1 + \ldots + (x^T x_n)x_n \), show that
\[
x_1 x_1^T + \ldots x_n x_n^T = I.
\]
(b) By writing \( Ax = AIx \), show that
\[
A = \lambda_1 x_1 x_1^T + \ldots \lambda_n x_n x_n^T.
\]
(c) Show that if \( \lambda_1 \neq 0, \ldots, \lambda_n \neq 0 \), then
\[
A^{-1} = \frac{1}{\lambda_1} x_1 x_1^T + \ldots + \frac{1}{\lambda_n} x_n x_n^T.
\]

**Exercise 6 [Power of a matrix]** Consider a real symmetric matrix \( A \in \mathbb{R}^{n \times n} \) with eigenvectors \( x_1, \ldots, x_n \) and eigenvalues \( \lambda_1, \ldots, \lambda_n \). Then, for any \( k \in \mathbb{N}_{\geq 1} \), the eigenvectors of \( A^k \) are \( x_1, \ldots, x_n \) and the eigenvalues \( \lambda_1^k, \ldots, \lambda_n^k \).
Exercise 7 [Trace of a matrix] The trace of a matrix is defined as

\[ \text{tr}(A) = \sum_{i=1}^{n} A_{ii}. \]

(a) Show that for any two matrices \( X \) and \( Y \), we have that \( \text{tr}(XY) = \text{tr}(YX) \).
(b) Show that for any real symmetric matrix \( A \), \( \text{tr}(A) = \sum_{i=1}^{n} \lambda_i \).

Exercise 8 [Determinant of symmetric matrix] Consider a real symmetric matrix \( A \). Show that

\[ \det(A) = \prod_{i=1}^{n} \lambda_i. \]

Hint: Use the property that \( \det(AB) = \det(A) \cdot \det(B) \).

2 Graph matrices

2.1 Adjacency matrix

Exercise 9 [Basic properties] Consider the adjacency matrix of an undirected graph.
(a) Show that \( \text{deg}(v_i) = \sum_{j=1}^{n} A_{ij} \).
(b) Show that the adjacency matrix of a \( d \)-regular graph has eigenvalue \( d \).

Exercise 10 [Counting paths] Consider the adjacency matrix \( A \) of a graph \( G \).
(a) Show that if \( (A^k)_{ij} > 0 \) for some integer \( k > 0 \) then there is a path of length \( k \) connecting \( i \) and \( j \).
(b) Show that \( (A^k)_{ij} \) also gives the number of paths connecting \( i \) and \( j \) with \( k \) hops.
(c) Interpret \( \text{tr}(A^k) > 0 \).

Exercise 11 [Bipartite graphs] Show that for any bipartite graph \( G \) with adjacency matrix \( A \), if \( \lambda > 0 \) is an eigenvalue then \( -\lambda \) is also an eigenvalue.

Exercise 12 [Converse for bipartite graphs]
(a) Argue that a graph with no odd length cycle is bipartite.
(b) Show that the statement “a graph has no odd length cycle” is equivalent to “for every odd \( k \), \( \text{tr}(A^k) = 0 \).”
(c) Deduce that if for every eigenvalue \( \lambda \) of \( A \), there is another eigenvalue \( -\lambda \), then the graph is bipartite.

Exercise 13 On Lecture 11/slide 10 (Example 1) we determined the spectrum of the adjacency matrix \( A \) for the complete graph (a.k.a. clique) of size 3. Here we would like to generalise this to any complete graph of size \( n > 3 \). Prove that the spectrum consists of eigenvalues \( n - 1 \) (with multiplicity 1) and \( -1 \) (with multiplicity \( n - 1 \)).

Exercise 14 [Perron-Frobenius] Let \( G \) by a connected graph with adjacency matrix \( A \) with eigenvectors \( x_1, \ldots, x_n \) and eigenvalues \( \lambda_1, \ldots, \lambda_n \), then show that
(a) \( \lambda_1 \geq -\lambda_n \),
(b) \( \lambda_1 > \lambda_2 \).
There exists an eigenvector $x_1$ which has all its entries $> 0$.

## 2.2 Laplacian matrix

**Exercise 15 [Factorisation]** Consider the unnormalised Laplacian matrix $L = D - A$ and the incident matrix $M \in \mathbb{R}^{n \times m}$ defined as $M_{ue} = 1_{u \in e}$ (i.e., indicates which edges contain which vertices). Show that

$$L = M^T M.$$

**Exercise 16** Consider an undirected, $d$-regular graph $G$ and the matrices $A_G$ and $L_G$.

(a) Show that the two matrices have the same eigenvectors.

(b) Describe the correspondence between their eigenvalues.

**Exercise 17** Consider a $d$-regular graph and its Laplacian matrix $L$.

(a) Using the quadratic form, show that for any vector $x \in \mathbb{R}^n$ (with $x \neq 0$),

$$\frac{x^T L x}{x^T x} \leq 2.$$

(b) Deduce that $\lambda_n \leq 2$.

Prove that for any $d$-regular graph, the largest eigenvalue of the Laplacian $L$ satisfies $\lambda_n \leq 2$.

**Exercise 18** Show that if $G$ is an undirected, $d$-regular, connected and bipartite graph, then the largest eigenvalue $\lambda_n$ of the Laplacian matrix satisfies $\lambda_n = 2$ (this proves one direction of the fourth statement in the Lemma from Lecture 11/slide 14).

**Exercise 19** Redo Exercise 19 without assuming that the graph is not $d$-regular.

**Exercise 20** Consider the transition matrix of a lazy random walks $\tilde{P} = (P + I)/2$ on a $d$-regular graph (here $I$ is the $n \times n$ identity matrix and $P$ is the transition matrix of a simple random walk).

(a) Using Exercise 17 and that the eigenvalues of $L$ are in $[0, 2]$, argue that the eigenvalues of $A$ are in $[-d, d]$.

(b) Prove that all eigenvalues of $\tilde{P}$ are non-negative.

## 3 Conductance

**Exercise 21 [Conductance of graphs]**

(a) Compute the conductance of the complete graph $K_n$.

(b) Compute the conductance of the cycle $C_n$.

(c) Compute the conductance of a path $P_n$.

(d) Compute the conductance of a 2D grid.

(e) (+) Compute the conductance of a 3D grid.

**Exercise 22**

(a) Prove that for every $n > 2$ there is an unweighted, undirected $n$-vertex graph with conductance 1.

(b) (+) Can you characterise all graphs with that property?
Exercise 23 Prove that for any $d$-regular graph with $n \to \infty$ being large, the conductance satisfies $\Phi(G) \leq \frac{1}{2} + o(1)$.

*Hint:* Use the probabilistic method to construct a set $S$ with the required conductance. First obtain bounds for $||S| - n/2|$ and then for $|E(S, S^c) - |E|/2|$.

Exercise 24 Consider an undirected, and $d$-regular graph $G = (V, E)$ with conductance $\Phi > 0$. In this exercise, you will show that the diameter of the graph is at most $O(\frac{\log n}{\Phi})$.

(a) Consider an arbitrary vertex $u \in V$ and let $S_0 := \{u\}$ and $S_i := B_{\leq i}(u)$, the set of nodes at a distance of at most $i$ from $u$. Show that for any $S_i$ with $|S_i| \leq n/2$, it holds that

$$|E(S_i, S_i^c)| \geq \Phi \cdot |S_i| \cdot d,$$

and that

$$|S_{i+1}| \geq |S_i| \cdot (1 + \Phi).$$

(b) Using that $\log(1 + z) \geq (1/2) \cdot z$ (for any $z \in [0,1]$), deduce that $|S_i| > n/2$ for $i \geq 2 \cdot \frac{\log n}{\Phi}$.

(c) Deduce that there is no pair of vertices at a distance $> \frac{4 \log n}{\Phi}$. 
