

# Randomised Algorithms Supervision 2

## Markov Chains

### 1 Properties of Markov Chains

**Exercise 1 [Chapman-Kolmogorov equations]** Consider a Markov chain  $(\rho^0, P)$  with state space  $\Omega$ . Let  $\rho^t \in \mathbb{R}$  be such that  $\rho_i^t = \Pr[X^t = i]$  for any  $i \in \Omega$ . Then, show that for any  $t \geq 0$ ,

$$\rho^t = \rho^0 \cdot P^t.$$

**Exercise 2 [Two-state MC]** Consider the Markov Chain with state space  $\Omega = \{0, 1\}$  and transition matrix

$$P = \begin{bmatrix} 1-p & p \\ q & 1-q \end{bmatrix},$$

for some  $p, q \in (0, 1)$ . By induction or otherwise, prove that

$$P^n = \frac{1}{p+q} \cdot \begin{bmatrix} q + p(1-p-q)^n & p - p(1-p-q)^n \\ q - q(1-p-q)^n & p + q(1-p-q)^n \end{bmatrix}.$$

Show that

$$\lim_{n \rightarrow \infty} P^n = \frac{1}{p+q} \cdot \begin{bmatrix} q & p \\ q & p \end{bmatrix}.$$

#### 1.1 Communicating states

**Exercise 3** Consider a Markov chain with transition matrix  $P$  and state space  $\Omega$ . Let  $G = (\Omega, E)$  be the graph where there is an edge  $(x, y) \in E$  iff  $P(x, y) > 0$ . State  $x$  is *accessible* from state  $y$  if there is a path from  $x$  to  $y$  in  $G$ . We say that  $i$  and  $j$  *communicate* (and write  $i \leftrightarrow j$ ) iff  $i$  is accessible from  $j$  and the other way round. Prove that for any  $i, j, k \in \Omega$

- $i \leftrightarrow i$  (**reflexive**).
- If  $i \leftrightarrow j$ , then also  $j \leftrightarrow i$  (**symmetric**).
- If  $i \leftrightarrow j$  and  $j \leftrightarrow k$ , then  $i \leftrightarrow k$  (**transitive**).

**Exercise 4 [Connection to Strongly Connected Components]** Consider a finite state Markov chain. Argue that the communicating classes are exactly the *strongly connected components* of the graph with vertices  $\Omega$  and edges  $uv \in E$  iff  $P_{uv}^t > 0$ .

**Exercise 5** (optional) How do the communicating classes of the Markov chain (or equivalently the induced strongly connected components of the graph) relate to *recurrent states*, i.e. states that will be visited infinitely often?

#### 1.2 Periodic/Aperiodic Markov Chains

The following exercises explore some core properties of periodicity of Markov chains, but they are not very easy to prove (relying on number theory) and can be skipped.

**Exercise 6** (++) Consider an irreducible finite Markov chain with state space  $\Omega$  and transition matrix  $P$ . Let  $T(x) = \{t \geq 1 : P^t(x, x) > 0\}$  be the set of all time steps that a Markov chain can return to  $x$ , having started at  $x$ . Then, the period of  $x$  is defined as  $\gcd T(x)$ .

Show that for any  $x, y \in \Omega$ , it holds that  $\gcd T(x) = \gcd T(y)$ .

**Exercise 7 (++)** For any aperiodic and irreducible Markov chain with state space  $\Omega$  and transition matrix  $P$ , there exists  $r \geq 0$  such that for any  $x, y \in \Omega$ ,

$$P^r(x, y) > 0.$$

**Exercise 8** Prove that in any periodic graph with period  $k$  it is possible to split the vertices into  $k$  disjoint sets  $C_0, \dots, C_{k-1}$  such that each edge  $(u, v) \in E$  satisfies  $u \in C_i$  and  $v \in C_{(i+1) \bmod k}$ .

### 1.3 Stationary distributions

**Exercise 9 [Non-unique stationary distribution]** Show that there is a non-irreducible Markov chain with more than one stationary distributions.

**Exercise 10 [Exact convergence]** Most Markov chains covered in this course never reach a stationary distribution exactly, but only get arbitrarily close. Can you find an irreducible Markov chain with  $n$  states such that for any starting state  $x$  there is an integer  $t$  such that  $P_x^t = \pi$ ?

**Exercise 11 [Singleton cut-set]** Show that for any finite state Markov chain with stationary distribution  $\pi$ , we have that for any state  $x \in \Omega$

$$\sum_{y \in \Omega: y \neq x} \pi_y P_{y,x} = \sum_{y \in \Omega: x \neq y} \pi_x P_{x,y}.$$

**Exercise 12 [Cut-set of Markov chain]** (+) Consider a finite, irreducible and aperiodic Markov chain. Then, for any  $S \subseteq \Omega$ , we have that,

$$\sum_{x \in S} \sum_{y \in S^c} \pi_x P_{xy} = \sum_{y \in S} \sum_{x \in S^c} \pi_y P_{yx}.$$

This means that the probability that the chain leaves the set  $S$  is equal to the probability that the chain enters  $S$ .

**Exercise 13 [Time-reversible Markov chains]** The following property is very useful for determining the stationary distribution of a Markov chain (see Exercises 20, 23, 43, 32).

Consider a finite irreducible Markov chain with transition matrix  $P$ . Assume that for some  $\tilde{\pi}$  with  $\sum_{x \in \Omega} \tilde{\pi}_x = 1$  and  $\tilde{\pi}_x \geq 0$ , we have that for any  $x, y \in \Omega$

$$\tilde{\pi}_x P_{xy} = \tilde{\pi}_y P_{yx}.$$

Then,  $\tilde{\pi}$  is the stationary distribution of the Markov chain.

**Extended Note 1 [Verifying the stationary distribution]** Exercise 13 gives us a way to quickly verify that a probability vector is the stationary distribution of an irreducible Markov chain: we just have to verify that it satisfies  $\pi_x P_{xy} = \pi_y P_{yx}$ , for any two states  $x$  and  $y$ .

**Exercise 14 [Time-reversal]** This exercise explains where the name time-reversal comes from (cf. Probability & Computing Ex. 7.13).

Consider a finite Markov chain with stationary distribution  $\pi$  and transition matrix  $P$ . Imagine starting the chain at time 0 and running it for  $m$  steps, obtaining the sequence  $X_0, \dots, X_m$ . Consider the states in reverse order,  $X_m, X_{m-1}, \dots, X_0$ .

- (a) Argue that given  $X_{k+1}$ , the state  $X_k$  is independent of  $X_{k+2}, \dots, X_m$ . Thus the reverse sequence is Markovian.
- (b) Argue that for the reverse sequence, the transition probabilities  $Q_{x,y}$  are given by

$$Q_{x,y} = \frac{\pi_y P_{y,x}}{\pi_x}.$$

- (c) Prove that if the original Markov chain is time reversible, so that  $\pi_x P_{x,y} = \pi_y P_{y,x}$ , then  $Q_{x,y} = P_{x,y}$ .
- (d) Based on your answer in (c), argue about the name *time-reversible*.

**Exercise 15 [Ehrenfest Chain]** Recall the Ehrenfest Chain defined on **Lecture 4 slide 3**.

- (a) Draw the Ehrenfest Chain for  $d = 5$ .
- (b) What is the stationary distribution of the Ehrenfest Chain?
- (c) Does the Ehrenfest Chain converge to the stationary distribution?

**Exercise 16** Let  $X_n$  be the sum of  $n$  independent rolls of a fair die. Show that for any  $k \geq 2$ ,

$$\lim_{n \rightarrow \infty} \Pr [X_n \text{ is divisible by } k] = \frac{1}{k}.$$

**Exercise 17** Consider a Markov chain on the states  $\Omega = \{0, 1, \dots, n\}$ , where for  $i < n$  we have  $P_{i,i+1} = 1/2$  and  $P_{i,0} = 1/2$ . Also,  $P_{n,n} = 1/2$  and  $P_{n,0} = 1/2$ . This process can be viewed as a random walk on a directed graph with vertices  $\{0, 1, \dots, n\}$ , where each vertex has two directed edges: one that returns to 0 and one that moves to the vertex with the next higher number (with a self-loop at vertex  $n$ ).

- (a) Find the stationary distribution of this chain.
- (b) (optional) What does this say about random walks in directed graphs?

**Exercise 18** Let  $(X_n)_{n \geq 1}$  be a Markov chain on the states  $\{0, 1, 2\}$  with transition matrix

$$P = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1-p & p \\ 1-p & p & 0 \end{bmatrix},$$

where  $p \in (0, 1)$ .

- (a) Draw the state space diagram for the Markov Chain  $X_n$ .
- (b) Explain why  $X_n$  is an irreducible and aperiodic Markov Chain.
- (c) Determine the stationary distribution of the Markov Chain.

[Source: [2017P6Q8]]

**Exercise 19** Attempt [2022P8Q12 (a),(b)(i)].

**Exercise 20 [Top-to-Random]** Consider the Top-to-Random shuffle defined on **Lecture 4 slide 23**.

- (a) Define the Markov chain corresponding to the shuffle.
- (b) Argue that the Markov chain is irreducible.
- (c) Argue that the Markov chain is aperiodic.

- (d) Find the stationary distribution of the chain.

**Exercise 21 [Riffle shuffle]** Consider the riffle shuffle defined on **Lecture 4 slide 24**.

- (a) Define the state space of the Markov chain.
- (b) (+) Consider the riffle operation. Given two decks of cards  $A$  and  $B$  with  $a$  and  $b$  cards, at each step, the next card is chosen from  $A$  with probability  $\frac{a}{a+b}$  and otherwise from  $B$ . Prove that when starting with  $n$  cards in total, drawing  $n$  cards using the above operation results into a uniform distribution over all permutations such that the subsequences of cards in  $A$  (and in  $B$ , respectively) are ordered increasingly.
- (c) Deduce the transition probabilities of the Markov chain.
- (d) Argue that the Markov chain is irreducible.
- (e) Find the stationary distribution of this chain.

## 1.4 Omitted proofs

**Further Reading 1 [Theorems for finite MCs]** This paper contains relatively short proofs for the major theorems for finite Markov chains: (i) existence, (ii) uniqueness and (iii) convergence.

**Further Reading 2 [2D and 3D Random walks]** See Lecture 6 in Part IB Markov Chains course from the Maths department, to find out why a drunk person finds his way home, but not a drunk bird.

## 2 Classical Markov Chains

**Exercise 22 [Birth-Death chains]** The *birth-death* Markov chains have state space  $\Omega = \mathbb{N}$  and transition probability matrix  $P$  satisfying

$$\begin{aligned}P_{i,i-1} &= p \quad \text{for } i = 1, 2, \dots \\P_{i,i+1} &= 1 - p \quad \text{for } i = 0, 1, \dots \\P_{0,0} &= p,\end{aligned}$$

for some parameter  $p \in (0, 1)$ .

- (a) Draw the first 3 states of the Markov chain. Why is it called a *birth-death* MC?
- (b) Show that the chain is irreducible.
- (c) Show that the chain is aperiodic.
- (d) For any  $p > 1/2$ , find the stationary distribution.

[Source: [2009P4Q10 (a)]]

**Exercise 23 [Bernoulli-Laplace Chain]** Consider two urns one with  $j$  balls and the other with  $k$  balls, having in total  $r$  red balls (and the other being red). In each step, we select uniformly at random one ball from each urn and swap them.

- (a) Design a Markov chain for  $X$  the number of red balls in urn 1, giving the state space and the transition matrix.
- (b) Prove that the chain is irreducible.
- (c) Prove that the chain is aperiodic.

(d) (+) Using Exercise 13 or otherwise, prove that the stationary distribution  $\pi$  is given by

$$\pi_x = \frac{\binom{r}{x} \cdot \binom{j+k-r}{k-x}}{\binom{j+k}{k}}.$$

**Exercise 24 [Gambler's ruin]** In the Gambler's ruin game there are two players  $A$  and  $B$ , starting with fortunes  $a$  and  $b$  respectively. In each step player  $A$  wins 1 with probability  $p$  and loses 1 with probability  $1 - p$ . Let  $X^t$  be the amount that player  $A$  has lost after  $t$  rounds (so  $X^0 = 0$ ). Player  $A$  will get broke when  $X^t = -a$ , while player  $B$  will get broke when  $X^t = b$ . In this exercise, we are interested in the expected length  $\ell(x)$  of a game starting with player  $A$  having lost  $x$  games.

(a) Define the Markov chain for this problem.

(b) Show that for  $p = 1/2$  and  $\ell(x)$  satisfying for any  $-a < x < b$ ,

$$\ell(x) = \frac{1}{2}\ell(x-1) + \frac{1}{2}\ell(x+1) + 1.$$

(c) Show that for  $p = 1/2$ ,

$$\ell(x) = (x+a) \cdot (b-x).$$

(d) (+ optional) Find a similar recurrence relation for  $p \neq 1/2$ , and deduce the expected length.

## 3 Applications

### 3.1 Algorithms

**Exercise 25 [2-SAT]** Prove rigorously the claim made in lecture that the expected time for RAND 2-SAT to find a given solution is at most the hitting time  $h(0, n)$  of the random walk on a path.

**Exercise 26 [3-SAT]**

(a) Design an algorithm for the 3-SAT problem similar to the one for 2-SAT described in the lectures.

(b) Let  $h_j$  be the number of expected steps to output a solution given that  $j$  of the literals are correct. Then, show that  $h_j$  satisfies

$$\begin{aligned} h_n &= 0, \\ h_j &= \frac{2}{3}h_{j-1} + \frac{1}{3}h_{j+1} + 1, \quad 1 \leq j \leq n-1, \\ h_0 &= h_1 + 1. \end{aligned}$$

(c) Prove by induction or otherwise that  $h_j = 2^{n+2} - 2^{j+2} - 3(n-j)$ .

(d) Does the resulting algorithm have a good running time?

**Exercise 27 [ $k$ -SAT]**

(a) Generalise the algorithm for the 2-SAT problem to  $k$ -SAT.

(b) (+) What is the expected running time as a function of  $k$ ? How does this compare to the brute force algorithm?

**Exercise 28 [Faster 3-SAT]** (+) Read 176-178 in "Probability and Computing" to see how to improve the expected running time for the 3-SAT algorithm (of Exercise 26) to  $\mathcal{O}(n^{3/2} \cdot (4/3)^n)$  time.

**Exercise 29 [3-Colourability]** A *colouring* of a graph is an assignment of a colour to each of its vertices. A graph is  $k$ -colourable if there is a colouring of the graph with  $k$  colours such that no two adjacent vertices have the same colour. Let  $G$  be a 3-colourable graph.

- (a) Show that there exists a colouring of the graph with two colours such that no triangle is monochromatic. (A triangle of a graph  $G$  is a subgraph of  $G$  with three vertices which are all adjacent to one another).
- (b) Consider the following algorithm for colouring the vertices of  $G$  with two colours so that no triangle is monochromatic. The algorithm begins with an arbitrary 2-colouring of  $G$ . While there are any monochromatic triangles in  $G$ , the algorithm chooses one such triangle and changes the colour of a randomly chosen vertex of that triangle. Derive an upper bound on the expected number of such recolouring steps before the algorithm finds a 2-colouring with the desired property.

**Exercise 30 [Reachability in log-space]** The reachability problem is the following: Given an undirected graph  $G = (V, E)$  and a pair of vertices  $s, t \in V$ , we want to determine if there is a path from  $s$  to  $t$  in  $G$ .

- (a) Argue that the standard DFS/BFS solution requires  $\Theta(n)$  space.
- (b) Argue that running a random walk with at most  $n^k$  (with constant  $k$ ) steps, requires  $\Theta(\log n)$  space.
- (c) Design an algorithm based on random walks which is able to answer the reachability problem with high probability.

**Exercise 31 [Randomised perfect bipartite matching]** In Part IA algorithms, you learnt about solving the maximum flow problem using the Ford-Fulkerson algorithm. You also learnt how to apply the algorithm to find a perfect matching in a bipartite graph in  $O(mn)$  time.

A recent result states that if you find the flow paths in Ford-Fulkerson using random walks (instead of using just BFS), then you can find a perfect matching in  $O(n \log n)$  time. You can read about the details here.

## 3.2 Sampling

**Exercise 32 [MCMC]** In this exercise, you will show that modifying the random walk by adding self-loops with appropriate probability, we obtain a Markov chain with the uniform stationary distribution. In particular, given a graph  $G = (V, E)$  and letting  $N(u)$  be the set of neighbours of  $u \in V$ , show that the Markov chain with transition matrix

$$P_{u,v} = \begin{cases} 1/M & \text{if } u \neq v \text{ and } v \in N(u), \\ 0 & \text{if } u \neq v \text{ and } v \notin N(u), \\ 1 - N(u)/M & \text{if } u = v, \end{cases}$$

for any  $M \geq |V|$  has the uniform stationary distribution. *Hint: You may want to use Exercise 13.*

**Exercise 33 [Sampling an independent set]** In this exercise, you will analyse the Markov chain algorithm for sampling an independent set (**Lecture 4 slide 26**).

- (a) Define the Markov chain for this setting.
- (b) Argue that the chain is irreducible. *Hint: Is there a state that can be reached from all states?*
- (c) Argue that the chain is aperiodic.
- (d) Using Exercise 32, argue that the uniform distribution is the stationary distribution of the Markov chain.

**Exercise 34 [Metropolis Algorithm]** In this exercise, we will generalise Exercise 32, so that given a connected graph  $G = (V, E)$  and a distribution  $\pi$  over  $V$ , we create a Markov chain over  $V$  to have  $\pi$  as its stationary distribution.

Show that the chain with transition matrix

$$P_{u,v} = \begin{cases} (1/M) \cdot \min(1, \pi_v/\pi_u), & \text{if } u \neq v \text{ and } v \in N(u), \\ 0, & \text{if } u \neq v \text{ and } v \notin N(u), \\ 1 - \sum_{v \neq u} P_{u,v}, & \text{if } u = v. \end{cases}$$

Why would we want to do this?

**Exercise 35** Combining Exercises 33 and 34, design a Markov chain whose stationary distribution is the exponential distribution over the independent sets  $\mathcal{I}$  of a graph  $G = (V, E)$ , where for any  $I \in \mathcal{I}$

$$\pi_I = \frac{e^{\lambda|I|}}{\sum_{I' \in \mathcal{I}} e^{\lambda|I'|}},$$

for any  $\lambda \in \mathbb{R}$ . What is the main advantage of this algorithm?

## 4 Random walks

### 4.1 Properties

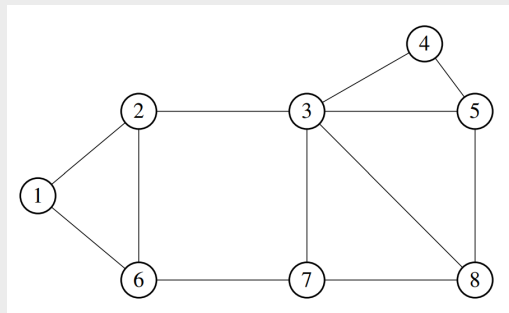
**Exercise 36 [Periodicity]**

- Prove that a simple random walk on a graph is periodic if the graph  $G$  is bipartite.
- (+) Can you also prove that the random walk is aperiodic if  $G$  is not bipartite?

**Exercise 37 [Stationary distribution]**

- Verify that  $\pi(u) = \frac{\deg(u)}{2|E|}$  is a stationary distribution of a simple random walk.
- Verify that the lazy random walk has the same stationary distribution.
- Does this hold for any graph?

**Exercise 38 [Concrete Example]** Suppose that  $G$  has eight vertices and undirected edges as shown in the figure below.



- Find the stationary distribution for the Markov Chain on  $G$ .
- Determine the relative proportions of time spent at each of the eight vertices.

[Source: [2007P4Q5 (c)]]

## 4.2 Variation distance

**Exercise 39** Prove that the total variation distance between two distributions over a finite probability space  $\Omega$  is a value in  $[0, 1]$ .

**Exercise 40** Consider a finite Markov chain  $(\mu, P)$  with state space  $\Omega$ . Show that

$$\|P_\mu^t - \pi\|_{TV} \leq \max_{x \in \Omega} \|P_x^t - \pi\|_{TV}$$

**Exercise 41** Let  $P$  be a transition matrix of a Markov chain with state space  $\Omega$  and  $\mu$  and  $\nu$  be two probability distributions on  $\Omega$ . Prove that

$$\|\mu P - \nu P\|_{TV} \leq \|\mu - \nu\|_{TV}$$

**Exercise 42** Consider a state space  $\Omega$  and for any probability  $p$  distribution over  $\Omega$ , define for any set  $A \subseteq \Omega$ ,  $p(A) = \sum_{\omega \in A} p(\omega)$ . Then, for probability distributions  $\mu$  and  $\nu$

$$\|\mu - \nu\|_{TV} = \max_{A \subseteq \Omega} |\mu(A) - \nu(A)|.$$

**Exercise 43** Consider the Ehrenfest Markov Chain  $P$  with state space  $\Omega = \{0, 1, \dots, d\}$ , and assume that the chain starts from state 0.

- Define a Markov chain  $Q$  with state space  $\Omega' = \{0, 1\}^d$ , where  $x \in \Omega'$  gives the colours of the particles.
- Relate the stationary distribution  $\pi_P$  to  $\pi_Q$ .
- In the Markov chain of (a), show that starting from state 0, then for any  $x_1, x_2 \in \Omega'$  with the same number of 1s, satisfy

$$Q_{0, x_1}^t = Q_{0, x_2}^t.$$

- Can you relate the variation distance  $\|P_0^t - \pi_P\|$  in the Ehrenfest chain to the variation distance  $\|Q_{0^d}^t - \pi_Q\|$  in the hypercube? *Hint: Use some symmetry argument.*

## 4.3 Mixing times

**Exercise 44** Attempt [2022P8Q12 (b)(ii),(iii)].

**Exercise 45** This question asks you to prove lower bounds on the mixing time of some lazy random walks on graphs.

- Let  $G = (V_1 \cup V_2, E)$  be a graph made of two disjoint complete graphs of  $n$  vertices, supported respectively on  $V_1$  and  $V_2$ , connected by a single edge. This is called the *Barbell graph*. Consider a lazy random walk on  $G$ . Prove that  $t_{mix}(1/4) = \Omega(n^2)$ .
- Suppose now we add  $s < n$  edges to the Barbell graph, where each edge has one endpoint in  $V_1$  and the other endpoint in  $V_2$ . What happens to  $t_{mix}(G)$ ?
- Consider now a version of the Barbell graph where  $|V_1| = n$ ,  $|V_2| = \lfloor \log n \rfloor$  and there exists only one edge between  $V_1$  and  $V_2$ . What is the mixing time of this graph?



## 4.4 Cover times

**Extended Note 2 [Cover times]** The *cover time* of a random walk is defined as the expected time for a random walk to visit all vertices of a graph, starting from the worst possible vertex.

**Exercise 46 [Cover time of clique]** Analyse the cover time of a simple random walk on the complete graph (clique), i.e., the graph where each pair of vertices is connected by an undirected edge. *Hint: Use coupon collector.*

**Exercise 47 [Cover time of path]** Consider a path  $P_n$  with vertex set  $\{0, 1, \dots, n\}$  for even  $n$ . Can you determine the cover time? *Hint: use Exercise 24.*

**Exercise 48 [Cover time of cycle]** What is the cover time of a cycle  $C_n$ ? *Hint: use Exercise 24.*

**Exercise 49 [Cover time using mixing time]** Consider any connected regular graph  $G = (V, E)$ . In this exercise, we will derive an upper bound on the cover time based on the mixing time  $t := t_{mix}(1/(2n))$  which is  $\mathcal{O}(t \cdot n \log n)$ .

(a) Let  $T_u := \min\{s \geq 1 : X_s = u\}$ . Show that starting from any vertex  $X_0 = u$ , we have that

$$\Pr[T_u \leq t \mid X_0 = u] \geq \frac{1}{2n}.$$

(b) Using probability amplification and the union bound, show that

$$\Pr\left[\max_{u \in V} T_u \leq 4nt \log n\right] \geq 1 - n^{-1}.$$

(c) Bound the expectation of the cover time.