Intro to Probability Example Sheet 3

1 Sum of distributions

Extended Note 1 [Computing the random variables] Consider two discrete independent random variables X and Y with pmfs f_X and f_Y. Then, we want to compute the pmf for the random variable $Z = X + Y$.
In order to do this we sum up the probabilities for all ways of making the sum z.

$$f_Z(z) = \mathbf{Pr}[Z = z] = \sum_{k=-\infty}^{\infty} \mathbf{Pr}[X = k, Y = z - k] = \sum_{k=-\infty}^{\infty} \mathbf{Pr}[X = k] \cdot \mathbf{Pr}[Y = z - k]$$

$$= \sum_{k=-\infty}^{\infty} f_X(k) \cdot f_Y(z - k).$$

Using this formula we can compute the pmf for Z.
For continuous random variables X and Y with pdfs f_X and f_Y, the formula becomes

$$f_Z(z) = \int_{k=-\infty}^{\infty} f_X(k) \cdot f_Y(z - k)\, dk.$$

This type of summation is also known as *convolution* and it is used in several places, like signal processing, computer vision or efficient computation (see this video if you would like to learn more).

Exercise 1 [Sum of Poisson r.vs.] Consider two independent Poisson r.vs. $X \sim \mathsf{Poi}(\mu)$ and $Y \sim \mathsf{Poi}(\lambda)$. Show that $Z = X + Y \sim \mathsf{Poi}(\mu + \lambda)$.

Exercise 2 [Sum of uniform distributions] Consider three independent uniform distributions $X_1, X_2, X_3 \in \mathcal{U}[0, 1]$.

(a) Determine the pdf for $S_2 = X_1 + X_2$.

(b) Determine the pdf for $S_3 = X_1 + X_2 + X_3$.

Exercise 3 Given the following pmf for random variables X and Y, compute the pmf for $Z = X + Y$.

	1	2	3	4
$\mathbf{Pr}[X = x]$	$\frac{1}{6}$	$\frac{2}{6}$	$\frac{1}{6}$	$\frac{2}{6}$
$\mathbf{Pr}[Y = y]$	0	$\frac{1}{4}$	$\frac{2}{4}$	$\frac{1}{4}$

2 Minimum/Maximum of random variables

Extended Note 2 [Computing the distribution function] Given two independent random variables X and Y with cummulative distribution functions F_X and F_Y, we want to compute the cummulative distribution function F_Z for $Z = \max\{X, Y\}$.
The main observation is to see that $\max\{X, Y\} \leq z$ iff both $X \leq z$ and $y \leq z$ (*Why?*). Then, we obtain

$$\begin{aligned} F_Z(z) &= \mathbf{Pr}[Z \leq z] = \mathbf{Pr}[\max\{X, Y\} \leq z] = \mathbf{Pr}[X \leq z, Y \leq z] = \mathbf{Pr}[X \leq z] \cdot \mathbf{Pr}[Y \leq z] \\ &= F_X(z) \cdot F_Y(z). \end{aligned}$$

Similarly for $Z = \min\{X, Y\}$ we have that

$$\begin{aligned} F_Z(z) &= \mathbf{Pr}[Z \leq z] = 1 - \mathbf{Pr}[Z > z] = 1 - \mathbf{Pr}[\min\{X, Y\} > z] = 1 - \mathbf{Pr}[X > z, Y > z] \\ &= 1 - \mathbf{Pr}[X > z] \cdot \mathbf{Pr}[Y > z] = 1 - (1 - F_X(z)) \cdot (1 - F_Y(z)). \end{aligned}$$

Exercise 4 [Minimum of uniform r.vs.] Consider n independent uniform random variables $X_1, \ldots, X_n \sim \mathcal{U}[0, 1]$.

(a) Determine the cummulative distribution function for $Z = \max\{X_1, \ldots, X_n\}$.

(b) Determine the probability density function for Z.

(c) Determine the expectation for Z.

Exercise 5 [Minimum of Exponential r.vs.] Consider two independent Exponential r.vs. $X \sim \mathsf{Exp}(\lambda)$ and $Y \sim \mathsf{Exp}(\mu)$. Find the cummulative distribution of $Z = \min\{X, Y\}$.

Exercise 6 [Minimum of geometric r.vs.] Consider two independent Geometric r.vs. $X \sim \mathsf{Geom}(p)$ and $Y \sim \mathsf{Geom}(q)$. Find the cummulative distribution of $Z = \min\{X, Y\}$.

3 Joint and marginal distributions

Exercise 7 The joint probability density function of X and Y is given by

$$f(x, y) = c \cdot (y^2 - x^2) \cdot e^{-y}, \quad -y \leq x \leq y, 0 < y < \infty.$$

(a) Find c.

(b) Find the marginal densities of X and Y.

(c) Find $\mathbf{E}[X]$.

Exercise 8 The joint probability density function of X and Y is given by

$$f(x, y) = e^{-x-y}, \quad 0 \leq x < \infty, 0 \leq y < \infty.$$

(a) Find $\mathbf{Pr}[X < Y]$.

(b) Find $\mathbf{Pr}[X < a]$.

Exercise 9 The joint probability density function of X and Y is given by

$$f(x, y) = 12xy(1 - x), \quad 0 < x < 1, 0 < y < 1.$$

(a) Are X and Y independent?

(b) Find $\mathbf{E}[X]$ and $\mathbf{E}[Y]$.

(c) Find $\mathbf{Var}[X]$ and $\mathbf{Var}[Y]$.

Exercise 10 Suppose that X and Y have a discrete joint distribution for which the joint PMF is defined as follows:

$$f(x,y) = \begin{cases} c|x+y|, & x=-1,0,1 \text{ and } y=-1,0,1 \\ 0, & \textit{otherwise}. \end{cases}$$

Determine:

(a) Determine c.

(b) Determine $\mathbf{Pr}[X=0, Y=1]$ and $\mathbf{Pr}[X=1]$.

(c) Determine $\mathbf{Pr}[|X-Y|<1]$.

Further Reading 1 [Further exercises] You can find more exercises with solutions here and here.

4 Computing the variance

Exercise 11 [Hats] There are n people taking their hats randomly. Let N be the total number of people that got the correct hat back.

(a) Show that $\mathbf{E}[N] = 1$.

(b) Show that $\mathbf{Var}[N] = 1$.

(c) Use Chebyshev's inequality to deduce bounds on N.

Exercise 12 [Max-Cut] In Part IA Algorithms, you saw the Min-Cut problem, where given a graph $G=(V,E)$ the goal is to find a subset $S \subseteq V$ such that the number of the edges crossing S and $V \setminus S$ is minimised. In this exercise, we will look at the problem of *maximising* the number edges crossing the cut.

Consider the algorithm that goes through the vertices one by one and adds it to S independently with probability $1/2$.

(a) Show that the exected size C of the cut produced is $|E|/2$. Argue that this is within a factor 2 of the optimal.

(b) Compute the $\mathbf{Var}[C]$.

(c) Use Chebyshev's inequality to deduce bounds on C.