Intro to Probability Example Sheet 2

1 Discrete random variables

1.1 Introductory

Exercise 1 A discrete random variable X has the following probability distribution:

x	0	1	2	3	4
$\mathbf{Pr}[X = x]$	$1/8$	$3k$	$k/6$	$1/4$	$k/6$

(a) Find the exact value of k.

(b) Compute $\mathbf{Pr}[0 < X < 4]$.

(c) Sketch the pmf.

(d) Sketch the cdf.

Exercise 2 The discrete random variable X has the following probability distribution:

$$\mathbf{Pr}[X = x] = \begin{cases} k/x, & x = 1, 2, 3, 4 \\ 0, & \text{otherwise} \end{cases}$$

(a) Compute the exact value of k.

(b) Compute $\mathbf{E}[X]$.

1.2 Expectation

Exercise 3 [Basic expectation properties] Derive the following properties for the expectation of a discrete random variable:

(a) Assuming $\mathbf{E}[X]$ exists, show that $\mathbf{E}[aX + b] = a\mathbf{E}[X] + b$, for any constants a and b.

(b) State the Law of Unconscious Statistician.

(c) The expectation of an rv taking only non-negative values, is non-negative.

Exercise 4 [Basic variance properties] Derive the following properties for the variance of a discrete random variable:

(a) $V[X] = \mathbf{E}\left[X^2\right] - (\mathbf{E}[X])^2$.

(b) Use Exercise **??** (c) to deduce that $V[X] \geq 0$.

(c) $V[aX + b] = a^2 V[X]$.

Exercise 5 [Independent random variables] Let X and Y be independent random variables,

(a) Show that $\mathbf{E}[XY] = \mathbf{E}[X]\mathbf{E}[Y]$, provided both expectations exist.

(b) Show that $\mathbf{E}[(X - \mathbf{E}[X])(Y - \mathbf{E}[Y])] = 0$.

(c) Show that $V[X + Y] = V[X] + V[Y]$.

Exercise 6 [Identically independently distributed] Let $X_1, \ldots, X_n$ be identically independently distributed random variables with mean μ and standard deviation σ.

(a) Compute $\mathbf{E}[\sum_{i=1}^{n} X_i]$.

(b) Compute $V[\sum_{i=1}^{n} X_i]$.

Exercise 7 Consider a random variable X with $\mathbf{E}[X] = 1$ and $V[X] = 5$. Compute

(a) $\mathbf{E}\left[(2+X)^2\right]$;

(b) $V[4+3X]$.

[Source: Ross P4.38]

1.3 Bernoulli distribution

Exercise 8 Let $X \sim \mathsf{Ber}(p)$ be a Bernoulli random variable

(a) Compute $\mathbf{E}[X]$.

(b) Compute $V[X]$.

1.4 Binomial distribution

Exercise 9 Using Exercise **??**, compute the mean and variance of the Binomial distribution.

Exercise 10 Let $X \sim \mathsf{Bin}(n,p)$ be a Binomial random variable. By considering the ratio $\mathbf{Pr}[X = k+1] / \mathbf{Pr}[X = k]$, show that the Binomial pmf is increasing and then decreasing. Explain the figures on slide 24, lecture 4.

1.5 Geometric distribution

Exercise 11 [Expectation of non-negative discrete rv]

(a) Let X be a discrete random variable that takes on only non-negative integer values. Then,

$$\mathbf{E}[X] = \sum_{i=1}^{\infty} \mathbf{Pr}[X \geq i]$$

(b) Consider $X \sim \mathsf{Geom}(p)$. Use (a) to derive the expectation of a geometric random variable.

Exercise 12 [Memoryless property] Let $X \sim \mathsf{Geom}(p)$. Show that for $n, m \in \mathbb{N}$,

$$\mathbf{Pr}[X > n+m \mid X > n] = \mathbf{Pr}[X > m].$$

Exercise 13 [Unique distribution with memoryless property] (Optional) By consider $\mathbf{Pr}[N > n+1] / \mathbf{Pr}[N > n]$, show that any discrete distribution that satisfies the memoryless property is a geometric distribution.

Exercise 14 [Connection to uniform distribution] (Optional) Let X_i be n Bernoulli trials and $Y_n = \sum_{i=1}^{n} X_i$. Let K be the position of the first success. Show that $\mathbf{Pr}[K = j \mid Y_n = 1] = 1/n$.

Exercise 15 [Alternating tossing game] A coin has probability $p > 0$ of drawing heads. There are n players tossing a coin in round robin style: player 1 first, player 2 second, ..., player n, player 1 again, ...The first player to toss a head wins. Let W denote the winning player.

(a) Show that $\mathbf{Pr}[W = i] = \frac{p(1-p)^{i-1}}{1-(1-p)^n}$.

(b) Show that a player with a higher number has less chance of winning. Is this reasonable?

Exercise 16 [Odd one out] A coin has probability p of drawing heads. There are $k > 2$ players and in each round all players toss a coin. If all players have the same draw except for one, then the odd one out is eliminated and the rest of the players continue the game (until two players remain).

(a) Show that the probability of being a odd one out is

$$kp(1-p)^{k-1} + kp^{k-1}(1-p).$$

(b) Find the expected number of rounds for until only two players remain for $k = 4$ players and $p = 1/4$.

1.6 Poisson distribution

Exercise 17 [Basic properties of the Poisson distribution] Let $X \sim \mathsf{Po}(\lambda)$.

(a) Compute $\mathbf{E}[X]$.

(b) Show that for any random variable Y, $V[Y] = \mathbf{E}[Y(Y-1)] + \mathbf{E}[Y] - (\mathbf{E}[Y])^2$.

(c) Use (b) to derive the variance of X.

(d) Show that $\mathbf{E}\left[X^k\right] = \lambda \mathbf{E}\left[(X+1)^{k-1}\right]$.

Exercise 18 Derive the Poisson approximation of the Binomial distribution.

Exercise 19 Consider n coins, each of which independently comes up heads with probability p. Suppose that n is large and p is small, and let $\lambda = np$. Suppose that all n coins are tossed; if at least one comes up heads, the experiment ends; if not, we again toss all n coins, and so on. That is, we stop the first time that at least one of the n coins come up heads. Let X denote the total number of heads that appear. Which of the following reasonings concerned with approximating $\mathbf{Pr}[X = 1]$ is correct (in all cases, Y is a Poisson random variable with parameter λ)?

(a) Because the total number of heads that occur when all n coins are rolled is approximately a Poisson random variable with parameter λ,

$$\mathbf{Pr}[X = 1] \approx \mathbf{Pr}[Y = 1] = \lambda e^{-\lambda}.$$

(b) Because the total number of heads that occur when all n coins are rolled is approximately a Poisson random variable with parameter λ, and because we stop only when this number is positive,

$$\mathbf{Pr}[X = 1] \approx \mathbf{Pr}[Y = 1 \mid Y > 0] = \frac{\lambda e^{-\lambda}}{1 - e^{-\lambda}}.$$

(c) Because at least one coin comes up heads, X will equal 1 if none of the other $n-1$ coins come up heads. Because the number of heads resulting from these $n-1$ coins is approximately Poisson with mean $(n-1)p \approx \lambda$,

$$\mathbf{Pr}[X = 1] = \mathbf{Pr}[Y = 0] = e^{-\lambda}.$$

[Source: Ross T4.20]

1.7 Negative Binomial distribution

Exercise 20 [Basic properties] Let $X \sim \mathsf{NegBin}(r, p)$.

(a) Argue why the pmf of X is a valid pmf.

(b) Use Exercise **??** to derive $\mathbf{E}[X]$ and $V[X]$.

Exercise 21 [Banach match problem] At all times, a pipe-smoking mathematician carries 2 matchboxes—1 in his left-hand pocket and 1 in his right-hand pocket. Each time he needs a match, he is equally likely to take it from either pocket. Consider the moment when the mathematician first discovers that one of his matchboxes is empty. If it is assumed that both matchboxes initially contained N matches, what is the probability that there are exactly k matches, $k = 0, 1, \dots, N$, in the other box?

1.8 Hypergeometric distribution

Exercise 22 [Basic properties] Let $X \sim \mathsf{Hyp}(N, n, m)$.

(a) Argue why the pmf of X is a valid pmf.

(b) Use exercise 6 to derive $\mathbf{E}[X]$ and $V[X]$.

Exercise 23 A club contains 50 members; 20 are men and 30 are women. A committee of 10 members is chosen at random. Find each of the following:

(a) The probability mass function of the number of women on the committee.

(b) The mean and variance of the number of women on the committee.

(c) The mean and variance of the number of men on the committee.

(d) The probability that the committee members are all the same gender.

Exercise 24 Let $X \sim \mathsf{Hyp}(N, n, m)$ be a Hypergeometric random variable. By considering the ratio $\mathbf{Pr}[X = k+1] / \mathbf{Pr}[X = k]$, show that the Hypergeometric pmf is increasing and then decreasing.

2 Continuous random variables

2.1 Introductory

Exercise 25 A random variable has a probability density function given by

$$f(x) = \begin{cases} kx(2-x), & 0 \le x \le 2 \\ 0, & \text{otherwise} \end{cases}$$

(a) Compute k.

(b) Sketch the graph of f.

(c) Compute the cdf of the pdf.

(d) Compute $\mathbf{E}[X]$.

Exercise 26 The continuous random variable X has the probability density function

$$f(x) = \begin{cases} \frac{1}{6}x(1+x^2) & \text{for } 0 \le x \le 2 \\ 0 & \text{otherwise} \end{cases}$$

(a) Show that this is a pdf.

(b) Sketch the graph of f.

(c) Compute the mode of X.

(d) Find the mean of X.

(e) Find the median of X.

2.2 Uniform distribution

Exercise 27 [Basic properties] Let $X \sim \mathsf{Uni}(a, b)$.

(a) Show that the pdf of X is valid.

(b) Compute and sketch the cdf of X.

(c) Compute $\mathbf{E}[X]$.

(d) Compute $V[X]$.

2.3 Exponential distribution

Exercise 28 [Basic properties] Let $X \sim \mathsf{Exp}(\lambda)$.

(a) Show that the pdf of X is valid.

(b) Compute and sketch the cdf of X.

(c) Compute $\mathbf{E}[X]$.

(d) Compute $V[X]$.

Exercise 29 [Memoryless property] Let $X \sim \mathsf{Exp}(\lambda)$. Show that $\mathbf{Pr}[X > t + s \mid X > s] = \mathbf{Pr}[X > t]$ for every t and s.

Exercise 30 [Median] Let $X \sim \mathsf{Exp}(\lambda)$. Find the median of the distribution.

2.4 Normal distribution

Exercise 31 [Basic properties] Let $X \sim \mathcal{N}(\mu, \sigma)$.

(a) Compute $\mathbf{E}[X]$.

(b) Why do we use a lookup table for the cdf of the normal distribution?

(c) How would you use the lookup table for different values of μ and σ?

Exercise 32 [Normal distribution has a valid pdf] (Optional) In this exercise, you will show that the Normal distribution has a valid pdf.

(a) Show that

$$\left(\int_{-\infty}^{\infty} e^{-x^2} dx\right)^2 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-x^2} e^{-y^2} dx dy.$$

(b) By transforming to polar coordinates, show that the RHS is equal to π.

(c) Complete the proof that this is a valid pdf.