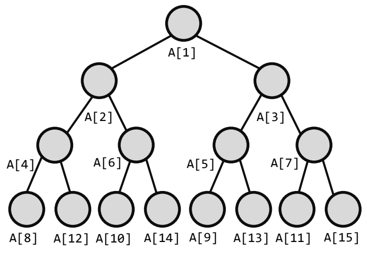
Enumerating entries in a functional array

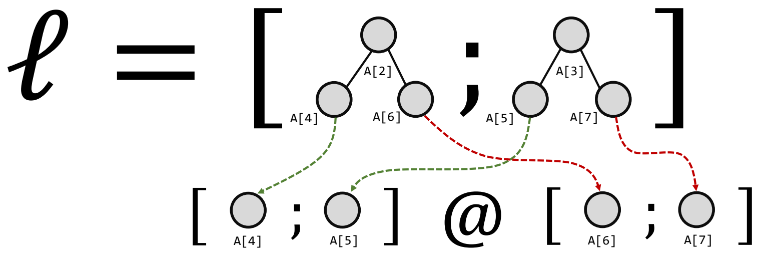
*In this short note, we look at the problem of enumerating the entries of a functional array in order. This was the topic of **[2015P1Q1 (c)]**, where a solution with time complexity $\mathcal{O}(n \log n)$ was presented, where n is the number of entries in the functional array, which in the worst-case is asymptotically as efficient as retrieving all indices and sorting them. Here, we will look at a solution that does this in $\mathcal{O}(n)$ time.*

1 Problem formulation and main observation

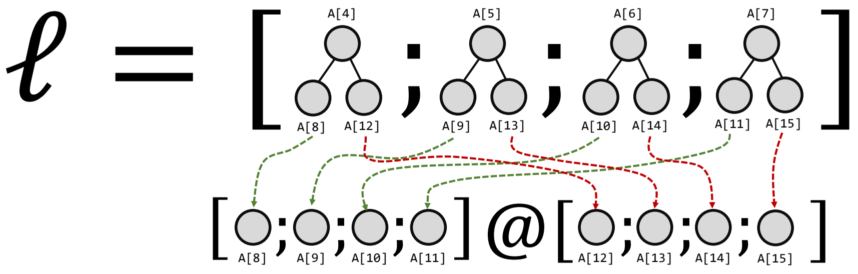
Consider the following functional array with 15 entries, where $A[i]$ indicates the value of the i-th entry in the functional array.



Note that the entries are not so important (it is mostly the indices that matter). Our goal is to design an algorithm that given this binary tree returns the list $[A[1]; A[2]; \ldots; A[14]; A[15]]$.
Main observation: Assuming that we have the nodes at level i sorted by their index in list ℓ, then we can generate the nodes at level $i+1$ sorted by their index, by appending the left children of nodes in ℓ and then the right nodes of the children in ℓ.
For example, for level 2, we have nodes $A[2]$ and $A[3]$ in order, so we first take the left nodes of $A[2]$ and $A[3]$ ($A[4]$ and $A[5]$ respectively), followed by the right nodes of $A[2]$ and $A[3]$ ($A[6]$ and $A[7]$ respectively).



Similarly, for level 3, we have,



Why does this observation hold? Note that the indices of the children of a node with index x at level i is given by $x + 2^{i-1}$ (left child) and $x + 2^{i}$ (right child). You can justify by looking at the binary representation of x. For example, if $x = 10 = 1010_2$, then the path to index 10 is given by 010_2, so the path to its left child is 10010_2 (which adds 2^3) and to its right child is 11010_2 (which adds 2^4). More generally for index x its left child holds index $x + 2^{i-1}$ and its right child holds index $x + 2^i$ where i is the depth of the node x.

2 Implementation

We are going to break down the implementation in the following:

1. Given the nodes at the current level sorted by index, create a list containing the nodes at the next level sorted by index (`get_next_layer`).
2. Recursively call the above function to create an enumeration of the elements of the functional array in order of their index (`get_in_order`).
3. Implement the filter function on the returned list (`filter_indices`). (what was originally asked in the question)

Before proceeding, try to implement these on your own.

Implementation of `get_next_layer`:

```
(* Some simple functions defined here to avoid clutter. *)
let get_left (Br(_, l, _)) = l;;
let get_right (Br(_, _, r)) = r;;
let get_value (Br(v, _, _)) = v;;

let get_next_layer cur =
   List.filter (fun x -> x <> Lf)
      (* The main part of the implementation is the following line.
         The filter is needed just in case there are leaf nodes. *)
      ((List.map get_left cur) @ (List.map get_right cur));;
```

Implementation of `get_in_order`:

```
let rec get_in_order = function
  [] -> []
| rt -> (List.map get_value rt) @ get_in_order (get_next_layer rt);;

get_in_order [root];;
```

Implementation of `filter_indices`:

```
let rec filter_indices f n = function
  [] -> []
| x::xs -> if f x then n :: filter_indices f (n+1) xs
           else filter_indices f (n+1) xs;;

filter_indices (fun x -> x mod 3 = 0) 0 [2;3;6;19;21;27;10;11];;
```

3 Verifying correctness

We will perform a simple test for enumeration using the function array implementation used in the lecture notes and a functional array with 25 elements.

```
(* Functional array *)

type 'a btree = Lf | Br of 'a * 'a btree * 'a btree;;

exception Subscript;;

let rec sub = function
| Lf, _ -> raise Subscript
| Br (v, t1, t2), 1 -> v
| Br (v, t1, t2), k when k mod 2 = 0 -> sub (t1, k / 2)
| Br (v, t1, t2), k -> sub (t2, k / 2);;

let rec update = function
| Lf, k, w ->
   if k = 1 then Br (w, Lf, Lf)
   else raise Subscript (* Gap in tree *)
| Br (v, t1, t2), k, w ->
   if k = 1 then Br (w, t1, t2)
   else if k mod 2 = 0 then Br (v, update (t1, k / 2, w), t2)
   else Br (v, t1, update (t2, k / 2, w));;


let root = Lf;;
let root = update (root, 1, 1);;
let root = update (root, 2, 2);;
let root = update (root, 3, 3);;
let root = update (root, 4, 4);;
let root = update (root, 5, 5);;
let root = update (root, 6, 6);;
let root = update (root, 7, 7);;
let root = update (root, 8, 8);;
let root = update (root, 9, 9);;
let root = update (root, 10, 10);;
let root = update (root, 11, 11);;
let root = update (root, 12, 12);;
let root = update (root, 13, 13);;
let root = update (root, 14, 14);;
let root = update (root, 15, 15);;

let root = update (root, 16, 16);;
let root = update (root, 17, 17);;
let root = update (root, 18, 18);;
let root = update (root, 19, 19);;
let root = update (root, 20, 20);;
let root = update (root, 21, 21);;
let root = update (root, 22, 22);;
let root = update (root, 23, 23);;
let root = update (root, 24, 24);;
let root = update (root, 25, 25);;

get_in_order [root];;

(*
- : int list =
[1; 2; 3; 4; 5; 6; 7; 8; 9; 10; 11; 12; 13; 14; 15; 16; 17; 18; 19; 20; 21;
 22; 23; 24; 25]
*)
```