Set Theory for Part IA Discrete Mathematics

Note: This handout contains several exercises and related past papers to the foundations of *Set Theory*.

The material in this handout appears in multiple books and other resources, so you will find a more complete treatment there. Most of this handout builds on Section 4.2 from Hamilton's "Numbers, sets and axioms: the apparatus of mathematics", which I recommend you read. Section 4.3 from the same book covers the construction of the natural numbers in ZFC. Chapter 1 constructs all the familiar sets $\mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C}$ from $\mathbb{N}$. These are outside the scope of the course, but motivate how set theory provides an axiomatisation of mathematics.

Intuition about sets

A set is a collection of items. The goal is to build as much as possible of mathematics using sets. This is what the different axiomatic theories try to do, succeeding at different degrees. The goal is for every mathematical object to be defined in terms of sets and all operations between mathematical objects to also be sets. The proofs in set theory may use the axioms and the operations of a logic system, like first-order logic.

As an example for how mathematical objects can be constructed just from sets, let's look at a high-level at the construction of the natural numbers. In Zermelo-Fraenkel (ZF) set theory, the natural numbers can be defined as the set consisting of: $\{\emptyset\}, \{\{\emptyset\}\}, \{\{\{\emptyset\}\}\}, \ldots$ (In Hamilton's book Section 4.3, the set $\emptyset, \emptyset \cup \{\emptyset\}, \emptyset \cup \{\emptyset\} \cup \{\emptyset \cup \{\emptyset\}\}, \ldots$ is used because some of the properties of natural numbers are simpler to prove.) Then for example, addition by one, e.g. $n+1$ is defined as $\{n\}$. From $\mathbb{N}$, one can construct $\mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C}$, relations and functions.

There are many great videos and introductory notes on the intuition of set theory, so this handout will not focus on this. It will mostly be a walk-through the ZF axioms along with solved problems.

One more thing about the axioms of set theory. Most of them are widely accepted, but several are still being argued. This is not just a theoretical debate without any practical consequences. For example, in measure theory (which is the the basis of continuous Probability), if we assume the axiom of choice, then there exists a subset of the real numbers that is not measurable, but if we don't assume it, then there does not exist one.

Zermelo-Fraenkel axioms

An initial list was started by Zermelo in 1908 [5] (see [4, p.199] for a translation) and was refined by Skolem [3] (see [4, p.290] for a translation) and Fraenkel in 1922 [1] (see [4, p.284] for a translation). The presentation here follows the exposition from the book "Numbers, sets and axioms".

In what follows there is no definition of 'set' nor for the symbol 'belongs to' (ϵ). Their properties are described by the axioms below. For convenience, we define the *subset* of a set,

Definition 1. For sets x and y, x is a subset of y (and y is a superset of x), denoted by $x \subset y$, if for every set z, $z \in x \Rightarrow z \in y$.

Note: You can think of $x \subseteq y$ as a shorthand for the logical formula $\forall z. z \in x \Rightarrow z \in y$. This quantifier formula can be manipulated by the logic axioms and rules just like any other logic formula.

(ZF1) (extensionality axiom) Two sets are equal iff they contain the same elements. More formally, for sets x and y,

$$x = y \Leftrightarrow (\forall z. z \in x \Leftrightarrow z \in y).$$

Example 1. Two sets are equal iff $x \subseteq y$ and $y \subseteq x$. (This is the anti-symmetry property for $\subseteq$)

Proof. A proof in ZF set theory proceeds by only manipulating logic statements. Let x and y be arbitrary sets, then

$$\begin{aligned} x = y &\Leftrightarrow (\forall z. z \in x \Leftrightarrow z \in y) \text{ (by ZF1)} \\ &\Leftrightarrow (\forall z. (z \in x \Rightarrow z \in y) \wedge (z \in y \Rightarrow z \in x) \text{ (by definition of } \Leftrightarrow) \\ &\Leftrightarrow (\forall z. (z \in x \Rightarrow z \in y)) \wedge (\forall z. (z \in y \Rightarrow z \in x)) \text{ (by properties of } \wedge \text{ and } \forall) \\ &\Leftrightarrow x \subseteq y \wedge y \subseteq x \text{ (by definition of subset)} \end{aligned}$$

□

Property 1. (Subset reflexivity) Prove that $\forall A. A \subset A$.

Proof. Let A be a set. We need to show that $a \in A \Rightarrow a \in A$. But this is a trivially true statement in logic (since $P \Rightarrow P$ is always true). □

Property 2. (Subset transitivity) Prove that $\forall A, B, C. (A \subset B \wedge B \subset C) \Rightarrow A \subset C$.

Proof. Let A, B and C be arbitrary sets. Assume that $A \subseteq B$ and $B \subseteq C$. By definition these assumptions mean that $\forall x.x \in A \Rightarrow x \in B$ (I) and $\forall y.y \in B \Rightarrow y \in C$ (II).

Assume $a \in A$, then using (I) for $x = a$ we have $a \in B$ and using (II) for $y = a$ we have that $a \in C$. Hence, we have shown that $\forall a.a \in A \Rightarrow a \in C$ or equivalently $A \subset C$. □

(ZF2) (null set axiom) There exists a set with no elements. More formally, $\exists S.\forall y.y \notin S$.

Note 1: $y \notin S$ is just shorthand for $\neg(y \in S)$.

Note 2: The empty set is denoted by $\emptyset$ or by $\{\}$.

Property 3. There is a unique empty set.

Proof. Assume that there are two empty sets e and e'. We will show that they are equal according to ZF1. We need to show $\forall z.(z \in e \Leftrightarrow z \in e')$ (I).

Because the two sets are empty, by definition we have $\forall z.(z \notin e)$ (II) and $\forall z.(z \notin e')$ (III). Showing that (II) and (III) imply (I), is an exercise in logic:

$$\begin{aligned} \forall z.(z \notin e) \wedge \forall z.(z \notin e') &\Rightarrow \\ \forall z.(\neg(z \in e) \wedge \neg(z \in e')) &\Rightarrow \\ \forall z.(z \in e \Leftrightarrow z \in e') & \text{ (since } \neg A \wedge \neg B \Rightarrow (A \Leftrightarrow B)) \end{aligned}$$

□

Property 4. Prove that for every set x, $\emptyset \subseteq x$.

Proof. We need to show that $\forall y.y \in \emptyset \Rightarrow y \in x$. Because, there is no set y that satisfies $y \in \emptyset$, the implication trivially holds. □

(ZF3) (pairing axiom) Given any sets x and y, there is a set u whose only elements are x and y. More formally,

$$\forall x.\forall y.\exists S.\forall t.(t \in S \Leftrightarrow (t = x \vee t = y)).$$

We denote this set by $\{x, y\}$.

Example 2. For any set z, there exists a set S such that S contains only z.

Proof. In the pairing axiom take $x = y = z$, so there exists a set S with $\forall t.(t \in S \Leftrightarrow (t = x \vee t = y))$ which is equivalent to $\forall t.(t \in S \Leftrightarrow t = x)$. □

Property 5. A basic property: $\{x, y\} = \{y, x\}$.

Proof. This follows from the commutativity of the $\vee$ logic operator:

$$\forall t.(t \in S \Leftrightarrow (t = x \vee t = y)) \Leftrightarrow \forall t.(t \in S \Leftrightarrow (t = y \vee t = x)).$$

□

We can now define ordered pairs, as follows

Definition 2. For sets x and y, the **ordered pair** $\langle x, y\rangle$ is defined as $\{\{x\}, \{x, y\}\}$.

Note: This is not the only way to define ordered pairs. In Exercise 28, pair is defined differently. The main (and perhaps only) property that we want the definition to satisfy is the following:

Why are we defining ordered pairs? Ordered pairs are used to define relations (and subsequently functions), where the order of the elements matters.

Property 6. For every sets a, b, c, d, $\langle a, b\rangle = \langle c, d\rangle \Rightarrow a = c \wedge b = d$.

Proof. Let $\langle a, b\rangle = \langle c, d\rangle$ i.e. $\{\{a\}, \{a, b\}\} = \{\{c\}, \{c, d\}\}$. Since $\{a\} \in$ RHS, there are two cases:

(a) $\{a\} = \{c\}$, in which case $a = c$. Now we consider two further cases:

- (i) $\{a, b\} = \{c, d\}$, in which case (since $a = c$), either $b = c = d (= a)$ or $b = d$.
- (ii) $\{a, b\} = \{c\}$, in which case $a = b = c$. Then, $\{c, d\} = \{a\}$ or $\{c, d\} = \{a, d\} = \{a\}$, so $d = a$.

(b) $\{a\} = \{c, d\}$, in which case $a = c = d$. In this case, $\{c, d\} = \{c\}$, so $\{a, b\} = \{c\}$. Hence, $a = b = c = d$.

In all cases, we get $a = c$ and $c = d$. □

> **(ZF4)** (union axiom) Given a set x (of sets), there is a set which has as its elements all elements of the elements of x. More formally,
>
> $$\forall x.\exists S.\forall z.z \in S \Leftrightarrow \exists y.y \in x \wedge z \in y.$$
>
> This set is known as the **big union** of $\bigcup x$.

Note: There will not be an intersection axiom, because intersection will follow by from the *separation axiom.*

Example 3. Determine $\bigcup\{\{1, 2, 8\}, \{2, 3, 7\}, \{7\}, \{6, 7\}\}$

Proof. This is just the elements that appear in at least one set. So, the big union is $\{1, 2, 3, 6, 7, 8\}$. □

In practice: In the algorithms course, you will learn about efficient ways of computing the big union of a set of sets (faster than the flatten method in the worst-case).

Definition 3. The **union** of the sets A and B, denoted by $A \cup B$ is defined by $\cup\{A, B\}$.

Property 7 (Key property)**.**

$$x \in A \cup B \Leftrightarrow x \in A \vee x \in B.$$

Property 8. (Commutativity of union) For sets A, B, show that $A \cup B = B \cup A$.

Proof.

$$x \in A \cup B \Leftrightarrow x \in A \vee x \in B \Leftrightarrow x \in B \vee x \in A \Leftrightarrow x \in B \cup A.$$

□

Property 9. (Associativity of union) For sets A, B, C, show that $A \cup (B \cup C) = (A \cup B) \cup C$.

Proof.

$$\begin{aligned} x \in A \cup (B \cup C) &\Leftrightarrow x \in A \vee x \in (B \cup C) \\ &\Leftrightarrow x \in A \vee (x \in B \vee x \in C) \\ &\Leftrightarrow (x \in A \vee x \in B) \vee x \in C \\ &\Leftrightarrow x \in A \cup B \vee x \in C \\ &\Leftrightarrow x \in (A \cup B) \cup C \end{aligned}$$

□

Property 10. The empty set is the identity element under union, i.e. for every set A, $A \cup \emptyset = \emptyset \cup A$.

Proof.

$$x \in A \cup \emptyset \Leftrightarrow x \in A \vee x \in \emptyset \Leftrightarrow x \in A,$$

since $x \in \emptyset$ is false for every x. □

Why do we not define big union in terms of the union operation? For finite sets this would indeed be equivalent. But for infinite sets, this process cannot be defined (and it is even harder for uncountable sets).

Example 4. Show that it is possible to use the pairing axiom to construct sets with n fixed elements: $x_1, \dots, x_n$. We use the notation $\{x_1, \dots, x_n\}$.

Proof. We can define these sets inductively for n. For $n = 1, 2$, it follows by the pairing axiom. Assume there is a set for $n-1$ elements, then $\{x_1, \dots, x_{n-1}\}$ is a set and x_n is a set. So, using the pairing axiom for these two sets, we get a set that contains $x_1, \dots, x_n$. □

(ZF5) (power set axiom) Given any set x, there is a set which has as elements all subsets of x. More formally,

$$\forall x. \exists S. \forall z. z \subseteq x \Leftrightarrow z \in S.$$

This set is known as the **power set** of x and it is denoted as $\mathcal{P}(x)$.

Now, we are going to introduce a set that consists of all sets that have a specific property p. However, this property cannot be unconstrained. Formally, the following observation is by Russell in 1901 [2],

Theorem 1. Defining sets with unconstrained formulas leads to a contradiction.

Proof. Let us consider the set z of all the sets x that satisfy $x \notin x$ (note that since all elements of a set are sets, it is a valid predicate). Using the notation below this is written as $z = \{x \notin x\}$. Since z is itself a set, it must either be that $z \in z$ or $z \notin z$.
If $z \notin z$, then z must be a member of z, since it consists of all sets with $x \notin x$. But this is a contradiction.
Otherwise, if $z \in z$, then z must not be a member of z. But this is again a contradiction.
So, such set cannot exist. □

So, we define the *well-formed formulas* as logical formulas consisting of propositions of the form $a \in b$ and $a = b$, where a and b are variables representing sets. More formally, R is a *well-formed formula*, iff P and Q are and R can be expressed as:

- $P \wedge Q$ standing for conjunction.
- $P \vee Q$ standing for disjunction.
- $P \Rightarrow Q$ standing for implication.
- $\forall u$ standing for the quantifier over all sets.
- $\exists u$ standing for the existential quantifier for sets.
- $\neg P$ standing for negation.
- $a \in b$ for some set variables a and b.
- $a = b$ for some set variables a and b.

You will see logic in more detail in Part IB Logic and Proof.
These well-formed formulas seem constraining, but this is needed to avoid Russel's paradox. Note however, that using these formulas it is possible to define intersection ($\cap$) and the subset relation ($\subseteq$), and thus make some formulas more succinct.

Now we are ready to introduce the separation axiom.

(ZF6) (separation axiom) Given any set x and a well-formed formula $p(\cdot)$, there exists a set consisting of all elements of x such that $p(\cdot)$ holds. More formally,

$$\forall x. \exists S. \forall z. z \in x \wedge p(z) \Rightarrow z \in S.$$

The set is denoted as $\{y \in x : p(y)\}$.

Note 1: This is rather a collection of axioms (called an axiom scheme), since we can instantiate it with any well-formed formula.

Note 2: By requiring $z \in x$, it is not possible to quantify over all sets, so we can avoid Russel's paradox.

Using the separation axiom, we can now define the intersection of two sets.

Definition 4. Let x and y be sets, then we define the **intersection** of these sets as $\{z \in x : z \in y\}$ (which exists by the separation axiom) and it is denoted by $x \cap y$.

More generally, we may define:

Definition 5. Let x be a set (of sets), then we define the **big intersection** of its elements as $\{z \in \bigcup x : \forall y. y \in x \Rightarrow y \in z\}$ (which exists by the separation axiom) and it is denoted by $\bigcap x$.

Definition 6. The **set difference** of sets x and y is defined as $\{z \in x : \neg(z \in y)\}$ (which exists by the separation axiom) and is denoted by $x \setminus y$.

We will see problems involving intersections, big intersections and set difference, in the problem section below.

Note 3: In a well-formed formula the quantifier could be asserting a property of the set that is being defined. It is not known if there are any contradictions that follow from this. (Read p.123 in Hamilton's book for a discussion on *impredicative* formulas).

The axioms that we have up to now, do not allow us to construct the sets that we would like to construct. Fraenkel proved that there is no way of constructing the set $\{x, \mathcal{P}(x), \mathcal{P}(\mathcal{P}(x)), \ldots\}$, because it is not constructed as a subset of a given set. This is why the following axiom was introduced:

(ZF7) (replacement axiom) Let $F(\cdot, \cdot)$ be a well-formed formula. Given any set x there is a set that consists of all elements u, such that there is $y \in x$ with $F(y, u)$ holds. More formally,

$$\forall x. \exists S. \forall u. ((\exists y. y \in x \wedge F(y, u)) \Rightarrow u \in S).$$

Now, let us see how we can use this axiom to create $\{x, \mathcal{P}(x), \mathcal{P}(\mathcal{P}(x)), \ldots\}$.

Example 5. Show that $\{x, \mathcal{P}(x), \mathcal{P}(\mathcal{P}(x)), \ldots\}$ is constructible.

Proof. This will be just an outline because we have not defined the natural numbers formally. The idea is that we can define $F(n, y)$ that takes a set and an integer n and checks if y is equal to $\mathcal{P}^n(x)$. $\square$

Actually, this axiom implies the previous axiom.

Theorem 2. Show that (ZF7) implies (ZF6).

Proof. For a predicate $p(\cdot)$ we can define, $F(y, u)$ to be $u = y \wedge p(x)$. Hence, the set of u that satisfy this relation consists exactly of those $y \in x$ that satisfy $p(y)$. $\square$

Note: This means that that (ZF6) is redundant. Some other simplifications of ZF theory are left as exercises in the "Further problems" section.

Another thing that we have not yet seen is how to construct infinite sets and gives a concrete example of such a set.

(ZF8) (infinity axiom) There is a set x such that $\emptyset \in x$, and such that for every set $u \in x$ we have $u \cup \{u\}$ also.

Note 1: This creates the set with elements: $\emptyset, \emptyset \cup \{\emptyset\}, \emptyset \cup \{\emptyset\} \cup \{\emptyset \cup \{\emptyset\}\}, \ldots$.

Note 2: It is possible to use the set $\{\}, \{\{\}\}, \{\{\{\}\}\}, \ldots$, but the one above is more standard.

The last core ZF axiom is one that asserts that by starting from a set x_0, picking an element $x_1 \in x_0$, then picking an element $x_2 \in x_3$ and so on, ought to stop at some finite point n. This is guaranteed by the following axiom:

(ZF9) (foundation axiom) Every non-empty set x contains an element which is disjoint from x.

Example 6. (optional) (ZF9) implies that there does not exist an infinite sequence of sets $x_0, x_1, x_2, \ldots$ that satisfy $x_{n+1} \in x_n$.

Proof. Consider the set $x = \{x_0, x_1, \ldots\}$ By (ZF9), there is $y \in x$ such that $y \cap x = \emptyset$. But $y = x_k$ for some k, so for $x_{k+1} \in x$ so $x_{k+1} \in x \cap y$. This leads to a contradiction. $\square$

Note: By taking $x_0 = x_1 = x_2 = \ldots = x$, we can see that it is not possible to have a set x with $x \in x$.

Although, (ZF9) is not required to prove that the elements of (ZF8) are unique , we can use it to get a simple argument,

Property 11. The elements of the set $\emptyset, \emptyset \cup \{\emptyset\}, \emptyset \cup \{\emptyset\} \cup \{\emptyset \cup \{\emptyset\}\}, \ldots$ are distinct and hence the set is infinite.

Proof. We will show that for any set x, the set $x \cup \{x\} \neq x$. Assume $x \cup \{x\} = x$, then $x \in x$ which is impossible. $\square$

Finally, we cover the axiom of choice, which is debated about whether it should be included in the core axioms of set theory (There are several researchers that do not accept it). ZF set theory together with the Axiom of Choice is called ZFC.

(AC) (axiom of choice) Given any (non-empty) set x whose elements are pairwise disjoint non-empty sets there exists a set which contains precisely one element from each set belonging to x.

Mathematical constructions using set theory

In the previous section we saw the ZFC axioms and how these are used to define some basic notions in set theory. In this section, we add more mathematical constructs (some of these will be the topic of another handout).

Definition 7. We tuples of n (natural number) elements, inductively, i.e. $(x_1, \ldots, x_{n+1}) = ((x_1, \ldots, x_n), x_{n+1})$.

In order to create a set of all pairs of elements (a, b) with $a \in A$ and $b \in B$, we need a superset of the pairs, so that we can apply the separation axiom. Since a pair is $\{\{a\}, \{a, b\}\}$, it is a member of $\mathcal{P}(\mathcal{P}(A \cup B))$ (note that this set may contain much more elements than we need). So, now we are ready for the definition:

Definition 8. The Cartesian product is defined as

$$A \times B = \{z \in \mathcal{P}(\mathcal{P}(A \cup B)) \mid \exists a. \exists b. a \in A \wedge b \in B \wedge Z = \langle a, b \rangle$$

Note: For n-tuples we can define this set as $X_1 \times X_2 \times \ldots \times X_{n+1} = (X_1 \times \ldots \times X_n) \times X_{n+1}$.

The Cartesian product allows us to define relations between the set A and B, which in turn allows us to define partial functions, functions, etc.

Taking the Cartesian product of a single element set $\{\ell\}$ and another set A is called **tagging**. The reason is that $\{\ell\} \times A = \{(\ell, a) \mid a \in A\}$. So, the resulting set can be interpreted as the elements of A with tag ℓ.

Using tagging, we can define the disjoint union of sets, which intuitively combines the two sets, but we can still detect the origin of the set (so it does not merge the elements that are the same).

Definition 9. The **disjoint union** of two sets A and B, denoted by $A \uplus B$, is the set $(\{1\} \times A) \cup (\{2\} \times B)$.

Of course, this definition can be generalised to

Definition 10. The disjoint union of sets $X_1, \ldots, X_n$, denoted by $\biguplus_{i=1}^{n} X_i$ is defined as the set

$$\biguplus_{i=1}^{n} X_i = (\{1\} \times X_1) \cup \ldots \cup (\{n\} \times X_n).$$

More problems and properties

Intersection

Property 12. (Commutativity of intersection) Given sets x and y, show that $x \cap y = y \cap x$.

Proof.
$$z \in x \cap y \Leftrightarrow z \in x \wedge z \in y \Leftrightarrow z \in y \wedge z \in x \Leftrightarrow z \in y \cap x.$$
Hence, by (ZF1) $x \cap y = y \cap x$. □

Property 13. (Associativity of intersection) Given sets x, y and z, show that $x \cap (y \cap z) = (x \cap y) \cap z$.

Proof.
$$\begin{aligned} k \in x \cap (y \cap z) &\Leftrightarrow k \in x \wedge k \in (y \cap z) \\ &\Leftrightarrow k \in x \wedge (k \in y \wedge k \in z) \\ &\Leftrightarrow (k \in x \wedge k \in y) \wedge k \in z \\ &\Leftrightarrow (k \in x \wedge k \in y) \cap z \end{aligned}$$
□

Property 14. For any sets x and y, show that $x \cap y \subseteq x$.

Proof.
$$k \in x \cap y \Rightarrow k \in x \wedge k \in y \Rightarrow k \in x.$$
□

Property 15. For any set x, show that $x \cap \emptyset = \emptyset$.

Proof. By the property above, $x \cap \emptyset \subseteq \emptyset$, so $x \cap \emptyset = \emptyset$. □

Property 16. (Distribution of union over intersection) For sets A, B, C show that $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.

Proof.
$$\begin{aligned} x \in A \cup (B \cap C) &\Leftrightarrow x \in A \vee x \in B \cap C \\ &\Leftrightarrow x \in A \vee (x \in B \wedge x \in C) \\ &\Leftrightarrow (x \in A \vee x \in B) \wedge (x \in A \vee x \in C)(x \in A \text{ (By distribution of } \vee \text{ over } \wedge) \\ &\Leftrightarrow (x \in A \cup B) \wedge (x \in A \cup C) \\ &\Leftrightarrow x \in (A \cup B) \cap (A \cup C) \end{aligned}$$
□

Property 17. (Distribution of intersection over union) For sets A, B, C show that $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

Proof.
$$\begin{aligned} x \in A \cap (B \cup C) &\Leftrightarrow x \in A \wedge x \in B \cup C \\ &\Leftrightarrow x \in A \wedge (x \in B \vee x \in C) \\ &\Leftrightarrow (x \in A \wedge x \in B) \vee (x \in A \wedge x \in C) \text{ (By distribution of } \wedge \text{ over } \vee) \\ &\Leftrightarrow (x \in A \cap B) \vee (x \in A \cap C) \\ &\Leftrightarrow x \in (A \cap B) \cup (A \cap C) \end{aligned}$$
□

Complement

Property 18. For any set A and complement A^c (with respect to U), $A \cup A^c = U$.

Proof.

$$\begin{aligned} x \in A \cup A^c &\Leftrightarrow x \in A \vee x \in A^c \\ &\Leftrightarrow x \in A \vee (x \in U \wedge x \notin A) \\ &\Leftrightarrow x \in U \vee (x \in U \wedge x \notin A) \text{ (since } A \subseteq U) \\ &\Leftrightarrow x \in U \end{aligned}$$

The second reverse step needs some clarification for why it holds. □

Property 19. For any set A and its complement A^c (with respect to U), $A \cap A^c = \emptyset$.

Proof.

$$\begin{aligned} x \in A \cap A^c &\Rightarrow x \in A \wedge x \in A^c \\ &\Rightarrow x \in A \wedge (x \in U \wedge x \notin A) \\ &\Rightarrow x \in A \wedge x \notin A \\ &\Rightarrow \textbf{false} \end{aligned}$$

□

Property 20. Show that $A \subseteq B \Rightarrow B^c \subseteq A^c$, where the complement is taken with respect to the set U.

Proof. Assume $A \subseteq B$. For $x \in U$, $x \in A \Rightarrow x \in B$. The contrapositive of this is $x \notin B \Rightarrow x \notin A$. Hence, $B^c \subseteq A^c$. □

Property 21. For any set A and its complement A^c (with respect to U), $(A^c)^c = A$.

Proof.

$$x \in (A^c)^c \Rightarrow x \in U \wedge x \notin A^c \Rightarrow x \in U \wedge \neg(x \in U \wedge x \in A^c) \Rightarrow x \in U \wedge (x \notin U \vee x \in A) \Rightarrow x \in U \wedge x \in A$$

□

Property 22. For any sets A, B in the universe U, $A^c \cup B^c = (A \cap B)^c$.

Proof.

$$\begin{aligned} x \in A^c \cup B^c &\Leftrightarrow (x \in U \wedge x \notin A) \vee (x \in U \wedge x \notin B) \\ &\Leftrightarrow x \in U \wedge (x \notin A \vee x \notin B) \\ &\Leftrightarrow x \in U \wedge \neg(x \in A \wedge x \in B) \text{ (Using De Morgan's law in logic)} \\ &\Leftrightarrow x \in U \wedge \neg(x \in A \cap B) \\ &\Leftrightarrow x \in U \wedge x \notin A \cap B \\ &\Leftrightarrow x \in (A \cap B)^c \end{aligned}$$

□

Property 23. For any sets A, B in the universe U, $A^c \cap B^c = (A \cup B)^c$.

Proof.

$$
\begin{aligned}
x \in A^c \cap B^c &\Leftrightarrow (x \in A^c) \wedge (x \in B^c) \\
&\Leftrightarrow (x \in U \wedge x \notin A) \wedge (x \in U \wedge x \notin B) \\
&\Leftrightarrow x \in U \wedge (x \notin A \wedge x \notin B) \\
&\Leftrightarrow x \in U \wedge \neg(x \in A \vee x \in B) \\
&\Leftrightarrow x \in U \wedge \neg(x \in A \cup B) \\
&\Leftrightarrow x \in U \wedge x \notin (A \cup B) \\
&\Leftrightarrow x \in (A \cup B)^c
\end{aligned}
$$

□

Exercise 1. Show that for sets A, B in the universe U, $(A \cup B) \cap (A^c \cup B) = B$.

Powerset

Example 7. Find the elements of $\mathcal{P}(\mathcal{P}(\mathcal{P}(\mathcal{P}(\emptyset))))$.

Proof. $\mathcal{P}(\emptyset) = \{\emptyset\}$, $\mathcal{P}^2(\emptyset) = \{\emptyset, \{\emptyset\}\}$, $\mathcal{P}^3(\emptyset) = \{\emptyset, \{\emptyset\}, \{\{\emptyset\}\}, \{\emptyset, \{\emptyset\}\}\}$,

$$
\begin{aligned}
&\mathcal{P}^4(\emptyset) = \{ \\
&\quad \emptyset, \\
&\quad \{\emptyset\}, \{\{\emptyset\}\}, \{\{\{\emptyset\}\}\}, \{\{\emptyset, \{\emptyset\}\}\}, \\
&\quad \{\emptyset, \{\emptyset\}\}, \{\emptyset, \{\{\emptyset\}\}\}, \{\emptyset, \{\emptyset, \{\emptyset\}\}\}, \{\{\emptyset\}, \{\{\emptyset\}\}\}, \{\{\emptyset\}, \{\emptyset, \{\emptyset\}\}\}, \{\{\{\emptyset\}\}, \{\emptyset, \{\emptyset\}\}\}, \\
&\quad \{\{\emptyset\}, \{\{\emptyset\}\}, \{\emptyset, \{\emptyset\}\}\}, \{\emptyset, \{\{\emptyset\}\}, \{\emptyset, \{\emptyset\}\}\}, \{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\}, \{\emptyset, \{\emptyset\}, \{\{\emptyset\}\}\}, \\
&\quad \{\emptyset, \{\emptyset\}, \{\{\emptyset\}\}, \{\emptyset, \{\emptyset\}\}\}\}
\end{aligned}
$$

□

Example 8. Show that $\bigcup \mathcal{P}(x) = x$.

Proof.

$$
\begin{aligned}
y \in \bigcup \mathcal{P}(x) = x &\Leftrightarrow \exists A. A \in \mathcal{P}(x) \wedge y \in A \\
&\Leftrightarrow \exists A. A \subseteq x \wedge y \in A \\
&\Leftrightarrow y \in x
\end{aligned}
$$

The reverse direction holds by considering the set $A = \{y\}$. Hence, the two sets are equal by the extensionality axiom. □

Example 9. Show that $\bigcap \mathcal{P}(x) = \emptyset$

Proof.

$$
\emptyset \in \mathcal{P}(x) \Rightarrow \bigcap \mathcal{P}(x) \subseteq \emptyset \Rightarrow \bigcap \mathcal{P}(x) = \emptyset.
$$

□

Example 10. For sets A and B,

$$
A \subseteq B \Rightarrow P(A) \subseteq P(B).
$$

Example 11. Find sets A and B such that the following statement is false:

$$
\mathcal{P}(A \cup B) \subseteq \mathcal{P}(A) \cup \mathcal{P}(B).
$$

Proof. Consider $A = \{1\}$ and $B = \{2\}$, then $\mathcal{P}(A \cup B) = \{\emptyset, \{1\}, \{2\}, \{1, 2\}\}$ and $\mathcal{P}(A) \cup \mathcal{P}(B) = \{\emptyset, \{1\}, \{2\}\}$. □

Example 12. For sets A and B,
$$\mathcal{P}(A) \cup \mathcal{P}(B) \subseteq \mathcal{P}(A \cup B).$$

Proof.

$$\begin{aligned} x \in \mathcal{P}(A) \cup \mathcal{P}(B) &\Leftrightarrow x \in \mathcal{P}(A) \vee x \in \mathcal{P}(B) \\ &\Leftrightarrow x \subseteq A \vee x \subseteq B \\ &\Leftrightarrow x \subseteq A \cup B \end{aligned}$$

□

Example 13. For sets A and B,
$$\mathcal{P}(A \cap B) = \mathcal{P}(A) \cap \mathcal{P}(B).$$

Proof.

$$\begin{aligned} x \in \mathcal{P}(A \cap B) &\Leftrightarrow x \subseteq A \cap B \\ &\Leftrightarrow x \subseteq A \wedge x \subseteq \cap B \\ &\Leftrightarrow x \in \mathcal{P}(A) \wedge x \in \mathcal{P}(B) \\ &\Leftrightarrow x \in \mathcal{P}(A) \cap \mathcal{P}(B) \end{aligned}$$

□

Cartesian Product

Property 24. For every set A, B, C, D,

$$(A \subseteq B) \wedge (C \subseteq D) \Rightarrow (A \times C) \Rightarrow (B \times D).$$

Proof. Assume that $(A \subseteq B) \wedge (C \subseteq D)$ holds, i.e. $A \subseteq B$ (I) and $C \subseteq D$ (II).

Consider $x \in A \times C$. By definition of the cross-product $x = (a, c)$ for $a \in A$ (III) and $c \in C$ (IV).

By (I) and (III), we have that $a \in C$. By (II) and (IV) we have that $c \in D$. Hence, $(a, c) \in C \times D$. □

Example 14. Find sets A, B, C, D such that the following does not hold,

$$(A \cup C) \times (B \cup D) \subseteq (A \times B) \cup (C \times D).$$

Proof. Consider $A = \{1\}$, $B = \{2\}$, $C = \{3\}$ and $D = \{4\}$. Then LHS $= \{(1, 2), (1, 4), (3, 2), (3, 4)\}$ and the RHS $= \{(1, 2), (3, 4)\}$. □

Property 25. For every set A, B, C, D,

$$(A \times B) \cup (C \times D) \subseteq (A \cup C) \times (B \cup D).$$

Proof. Assume $x \in (A \times B) \cup (C \times D)$, then $x \in A \times B$ or $x \in C \times D$.

Assume $x \in A \times B$, then $x = (a, b)$ for $a \in A$ and $b \in B$. So, $a \in A \cup C$ and $b \in B \cup D$. So, $x \in (A \cup C) \times (B \cup D)$.

Assume $x \in C \times D$, then ... (similar to the case above). □

Property 26. For every set A, B, C, D,

$$A \times (B \cup D) \subseteq (A \times B) \cup (A \times D).$$

Proof. Assume $x \in A \times (B \cup D)$, then $x = (a, c)$ for $a \in A$ and $c \in B \cup D$.

Assume $c \in B$, then $(a, c) \in (A \times B)$ and so $x \in (A \times B) \cup (A \times D)$.

Assume $c \in D$, then $(a, c) \in (A \times D)$ and so $x \in (A \times B) \cup (A \times D)$. □

Property 27. For every set A, B, C, D,

$$(A \times B) \cup (A \times D) \subseteq A \times (B \cup D).$$

Proof. Assume $x \in (A \times B) \cup (A \times D)$, then $x \in A \times B$ or $x \in A \times D$.

Assume $x \in A \times B$, then $x = (a, b)$ for $a \in A$ and $b \in B$. So, $b \in B \cup D$. Hence, $x \in A \times (B \cup D)$.

Assume $x \in A \times D$, ... (similarly to the above). □

Big Union

Exercise 2. (Hamilton 4.2.4 (iv)) Show that for any sets x, y, z, $\bigcup\{x, y, z\} = (x \cup y) \cup z$.

Exercise 3. (Hamilton 4.2.3) Let $x = \{\{\{y\}\}, \{\{y, \{z\}\}\}\}$. Find the elements of $\bigcup x$, $\bigcup\bigcup x$ and $\bigcup\bigcup\bigcup x$.

Example 15. (Hamilton 4.2.5) Prove that for any sets x and y, if $\bigcup x \neq \bigcup y$, then $x \neq y$. Is it the case that $\bigcup x = \bigcup y$ implies $x = y$?

Proof. It is easier to prove the contrapositive for the first, $x = y \implies \bigcup x = \bigcup y$.

No. Consider $x = \{\{1, 2\}, \{1, 3\}\}$ and $y = \{\{1, 2, 3\}\}$. □

Example 16. Show that for sets F_1 and F_2, $(\bigcup F_1) \cup (\bigcup F_2) = \bigcup(F_1 \cup F_2)$.

Proof.

$$\begin{aligned}
x \in (\bigcup F_1) \cup (\bigcup F_2) &\Leftrightarrow x \in (\bigcup F_1) \vee x \in (\bigcup F_2) \\
&\Leftrightarrow (\exists A. A \in F_1 \wedge x \in A) \vee (\exists A. A \in F_2 \wedge x \in A) \\
&\Leftrightarrow \exists A.((A \in F_1 \wedge A \in F_2) \wedge x) \\
&\Leftrightarrow \exists A.((A \in F_1 \cup F_2) \wedge x) \\
&\Leftrightarrow x \in \bigcup(F_1 \cup F_2)
\end{aligned}$$

□

Property 28. Prove that for all sets F, it holds that,

$$\forall U.(\bigcup F \subseteq U \Leftrightarrow (\forall X \in F. X \subseteq U)).$$

Proof. Consider an arbitrary set U.

($\Rightarrow$) Assume $\bigcup F \subseteq U$ (I). Assume $X \in F$. Consider $k \in X$, then $k \in U$ by (I). Hence, $X \subseteq U$ and the desired result follows.

($\Leftarrow$) Assume $\forall X \in F. X \subseteq U$ (II). Then consider $x \in \bigcup F$, then there exists $A \in F$, such that $x \in A$. But instantiating (II) with $X = A$, we get $A \subseteq U$. Hence, $x \in U$. □

Exercise 4. (Hamilton 4.3.1) Let x and y be sets. Find $\bigcup\langle x, y\rangle$, $\bigcup\bigcup\langle x, y\rangle$ and $\bigcup(x \times y)$.

Big Intersection

Property 29. Show that if $x \in F$, then $\bigcap F \subseteq x$.

Proof. Let $x \in F$,

$$\begin{aligned} y \in \bigcap F &\Rightarrow \forall A. A \in F \Rightarrow y \in A \\ &\Rightarrow x \in F \wedge y \in x \text{ (instantiating for } A = x) \\ &\Rightarrow x \in F \end{aligned}$$

□

Example 17. Show that for sets F_1 and F_2, $(\bigcap F_1) \cap (\bigcap F_2) = \bigcap (F_1 \cap F_2)$.

Proof.

$$\begin{aligned} x \in (\bigcap F_1) \cap (\bigcap F_2) &\Leftrightarrow x \in \bigcap F_1 \wedge x \in \bigcap F_2 \\ &\Leftrightarrow (\forall A. A \in F_1 \Rightarrow x \in A) \wedge (\forall A. A \in F_2 \Rightarrow x \in A) \\ &\Leftrightarrow \forall A. ((A \in F_1 \Rightarrow x \in A) \wedge (A \in F_2 \Rightarrow x \in A)) \\ &\Leftrightarrow \forall A. ((A \in F_1 \wedge A \in F_2) \Rightarrow x \in A) \\ &\Leftrightarrow \forall A. ((A \in (F_1 \cap F_2) \Rightarrow x \in A) \\ &\Leftrightarrow x \in \bigcap (F_1 \cap F_2) \end{aligned}$$

□

Example 18. For $F \subseteq \mathcal{P}(A)$ and $U = \{X \subseteq A \mid \forall S \in F. S \subseteq X\} \subseteq \mathcal{P}(A)$, show that $\bigcup F = \bigcap U$.

Proof. We will prove that the two sets are equal by showing that $\bigcup F \subseteq \bigcap U$ and then $\bigcap U \subseteq \bigcup F$.

(Direction: $\bigcap U \subseteq \bigcup F$) We will show that $\bigcup F \in U$. This will imply that (Property 29) that $\bigcap U \subseteq \bigcup F$. To show that $\bigcup F \in U$, $X = \bigcup F$ must satisfy $\forall S \in F. S \subseteq X$, which it does (Property 28.

(Direction: $\bigcup F \subseteq \bigcap U$) Consider $x \in \bigcup F$, then there exists $A \in F$ such that $x \in F$. Now we need to prove that $x \in \bigcap U$. To do this, we consider an arbitrary set $V \in U$. Then V satisfies $\forall S \in F. S \subseteq V$ (I). By instantiating (I) for $S = A$ we get $A \subseteq V$ and so $x \in V$. Hence, it holds for all V, so $x \in \bigcap U$. □

Example 19. For $F \subseteq \mathcal{P}(A)$ and $L = \{X \subseteq A \mid \forall S \in F. X \subseteq S\} \subseteq \mathcal{P}(A)$, show that $\bigcap F = \bigcup L$.

Proof. Again, we will prove that the two sets are equal by showing that $\bigcap F \subseteq \bigcup U$ and then $\bigcup L \subseteq \bigcap F$.

(Direction: $\bigcap F \subseteq \bigcup L$) We will show that $\bigcap F \in L$, so this will imply that $\bigcap F \in \bigcup L$. In order for $\bigcap F \in L$, $\bigcap F$ must satisfy $\forall S \in F. \bigcap F \subseteq S$, but this follows from the properties of big intersection.

(Direction: $\bigcup L \subseteq \bigcap F$) Consider $x \in \bigcup L$. Then there exists $X \in L$ that satisfies $x \in X$ (I) and $\forall S \in F. X \subseteq S$ (II). Combining (I) and (II), we get $\forall S \in F. x \in S$, so $x \in \bigcap F$. □

Disjoint set union

The key property that we use in all of the following properties is that

$$x \in A \uplus B \Leftrightarrow (\exists a. a \in A \wedge x = (1, a)) \vee (\exists b. b \in B \wedge x = (2, b))$$

Property 30. For any set A, B, C, D,

$$(A \subseteq B \wedge C \subseteq D) \Rightarrow A \uplus C \subseteq B \uplus D.$$

Proof. Assume $A \subseteq B \wedge C \subseteq D$, so $A \subseteq B$ (I) and $C \subseteq D$ (II).

Let $x \in A \uplus C$, so $x = (1, a)$ for $a \in A$ or $x = (2, c)$ $c \in C$.

Assume that $x = (1, a)$, then $a \in B$ by (I), so $x \in B \uplus D$.

Assume that $x = (2, c)$, then $c \in D$ by (II), so $x \in B \uplus D$. $\square$

Property 31. For any set A, B, C, D,

$$(A \cup B) \uplus C \subseteq (A \uplus C) \cup (B \uplus C).$$

Proof. Let $x \in (A \cup B) \uplus C$, so $x = (1, a)$ for $a \in A \cup B$ or $x = (2, c)$ for $c \in C$.

Assume $x = (1, a)$ for $a \in A \cup B$, then $a \in A$ or $a \in B$. If $a \in A$, then $x \in A \uplus C$. If $a \in B$, then $x \in B \uplus C$.

Assume $x = (2, c)$, then $x \in B \uplus C$.

So, in all three cases, $x \in (A \uplus C) \cup (B \uplus C)$. $\square$

Property 32.

$$(A \uplus C) \cup (B \uplus C) \subseteq (A \cup B) \uplus C.$$

Proof. Assume $x \in (A \uplus C) \cup (B \uplus C)$, then $x \in A \uplus C$ or $x \in B \uplus C$, so $x = (1, a)$, $x = (2, c)$, $x = (1, b)$ or $x = (2, c)$ for $a \in A$, $b \in B$ and $c \in C$.

For $x = (1, a)$ or $x = (1, b)$, since $a, b \in A \cup B$, $x \in (A \cup B) \uplus C$.

For $x = (2, c)$, $x \in (A \cup B) \uplus C$.

Hence, $x \in (A \cup B) \uplus C$. $\square$

Property 33.

$$(A \cap B) \uplus C \subseteq (A \uplus C) \cap (B \uplus C).$$

Proof. Let $x \in (A \cap B) \uplus C$, then $x = (1, a)$ for $a \in A \cap B$ or $x = (2, c)$ for $c \in C$.

Assume $x = (1, a)$ for $a \in A \cap B$, $a \in A$ and $a \in B$. Hence, $x \in A \uplus C$ and $x \in A \uplus C$. Hence, $x \in (A \cap B) \uplus C$.

Assume $x = (2, c)$ for $c \in C$, then $x \in A \uplus C$ and $x \in B \uplus C$. Hence, $x \in (A \cap B) \uplus C$. $\square$

Property 34.

$$(A \uplus C) \cap (B \uplus C) \subseteq (A \cap B) \uplus C.$$

Proof. Assume $x \in (A \uplus C) \cap (B \uplus C)$, then $x \in A \uplus C$ and $x \in B \uplus C$. Hence, $x = (1, a)$, or $x = (2, c)$ for $a \in A \cap B$ and $c \in C$.

Assume $x = (1, a)$ for $a \in A \cap B$, then $x \in (A \cap B) \uplus C$.

Assume $x = (2, c)$ for $c \in C$, then $x \in (A \cap B) \uplus C$.

Hence, $x \in (A \cap B) \uplus C$. $\square$

Set difference

Exercise 5. Show that for sets A and B in universe U, $A \setminus B = A \cap B^c$.

Exercise 6. Find sets A and B such that $A \setminus B \neq B \setminus A$.

Exercise 7. Show that $A \cap (B \setminus C) = (A \cap B) \setminus (A \cap C)$.

Symmetric difference

Definition 11. The symmetric difference of sets A and B is defined as $A \Delta B = (A \setminus B) \cup (B \setminus A)$.

Exercise 8. Show that for any sets A and B, $A \Delta B = B \Delta A$.

Exercise 9. Show that for any sets A, B and C, $A \Delta (B \Delta C) = (A \Delta B) \Delta C$.

Exercise 10. Show that for any set A, $A \Delta \emptyset = \emptyset \Delta A = A$.

Exercise 11. Show that for any sets A and its complement A^c (under the universe U), $A \Delta A^c = A^c \Delta A = U$.

Exercise 12. Show that $A \Delta B = A \cup B$ iff $A \cap B = \emptyset$.

Russel's paradox

Exercise 13. (Hamilton 4.2.9) Using (ZF6), derive a contradiction from the supposition that there is a set U containing all sets.

Exercise 14. (Hamilton 4.2.11) Derive a contradiction from the supposition that $\{y \mid x \subseteq y\}$ is a set for x being some fixed set.

Exercise 15. (Hamilton 4.2.12) Derive a contradiction from the supposition that $\{y \mid x \sim y\}$ is a set, where $\sim$ denotes cardinal equivalence. *Hint:* Assume that there is such set S and deduce that $\bigcup S$ contains every set.

The powerset Boolean Algebra

(optional) What is an Abstract Boolean Algebra?

Definition 12. An **abstract Boolean algebra** is defined as $(\mathcal{B}, 0, 1, +\cdot, \neg)$ with $0, 1 \in \mathcal{B}$ satisfying the following properties (where $x, y, z \in \mathcal{B}$):

- (**Identity laws**) $x + 0 = 0 + x = x$ and $1 \cdot x = x \cdot 1 = x$.
- (**Compliment laws**) $x + \neg x = 0$ and $x \cdot \neg x = 0$.
- (**Associative laws**) $x + (y + z) = (y + z) + x$ and $x \cdot (y \cdot z) = (x \cdot y) \cdot z$.
- (**Commutative laws**) $x + y = y + x$ and $x \cdot y = y \cdot x$.
- (**Distributive laws**) $x + (y \cdot z) = (x + y) \cdot (x + z)$ and $x \cdot (y + z) = x \cdot y + x \cdot z$.

As you have proven in other courses, the Boolean algebra is an abstract Boolean algebra for $0 = \text{false}$, $1 = \text{true}$, $\mathcal{B} = \{\text{false}, \text{true}\}$, $+ = \vee$, $\cdot = \wedge$ and $\neg x = \overline{x}$.

Derivation

The powerset Boolean algebra for U has $\mathcal{B} = \mathcal{P}(U)$, $0 = \emptyset$, $1 = U$, $+ = \cup$, $\cdot = \cap$ and $\neg x = x^c$ (the complement with respect to U). We proved in the problems above all of these properties, so the $(\mathcal{P}(U), \emptyset, U, \cup, \cap, (\cdot)^c)$ is a Boolean algebra.

An isomorphism

Maybe it is best to read this section once you have covered bijective functions.

Two discrete structures $(A, *_1)$ and $(B, *_2)$ are isomorphic if there exists a bijective function f that maps between the elements of A and B, such that:

$$\forall a_1 \in A. \forall a_2 \in A. f^{-1}(f(a_1) *_2 f(a_2)) = a_1 *_1 a_2.$$

Intuitively, this means that you can either perform the operation directly in $(A, *_1)$ or you can translate the elements to $(B, *_2)$ perform the operation there and then translate back to A.

In the lecture slides, you briefly discussed about the isomorphism from the (classic) Boolean Algebra $(\{\text{false}, \text{true}\}, \text{false}, \text{true}, \vee, \wedge, \neg(\cdot))$ to the powerset Boolean Algebra of a set with a single element $(\mathcal{P}(\{x\}) = \{\emptyset, \{x\}\}, \emptyset, \{x\}, \cup, \cap, (\cdot)^c)$. In this case, we have to prove that the isomorphism holds over three operations.

Exercise 16. Prove that this is indeed an isomorphism. *Hint:* See Example 3.9.1 in these notes.

Properties of Boolean Algebras

Exercise 17. Let $(\mathcal{B}, 0, 1, +, \cdot, \bar{\cdot})$ be a Boolean algebra, then it satisfies the following:

(a) (Idempotent Laws) $x + x = x$ and $x \cdot x = x$.

(b) (Domination Laws) $x + 1 = 1$ and $x \cdot 0 = 0$.

(c) (Absorption Laws) $(x \cdot y) + x = x$ and $(x + y) \cdot x = x$.

(d) $x + y = 1$ and $y \cdot x = 0$ iff $y = \overline{x}$.

(e) (Double complements Law) $\overline{\overline{x}} = x$.

(f) (De Morgan's Laws) $\overline{x \cdot y} = \overline{x} + \overline{y}$ and $\overline{x + y} = \overline{x} \cdot \overline{y}$.

(optional) Some more examples

You can find more examples of Boolean Algebras here.

Exercise 18. Show that $(\{1, 2, 3, 6\}, 1, 6, \mathrm{lcm}, \gcd, x \to 6/x)$ is a Boolean algebra.

Exercise 19. For $\mathcal{B} = \{\text{false}, \text{true}\}^k$ for some $k \in \mathbb{N}$ and the operators $\tilde{\vee}$, $\tilde{\wedge}$, $\tilde{\neg}$ being the element-wise $\vee$, $\wedge$ and $\neg$, show that these form a Boolean algebra and identify 0 and 1.

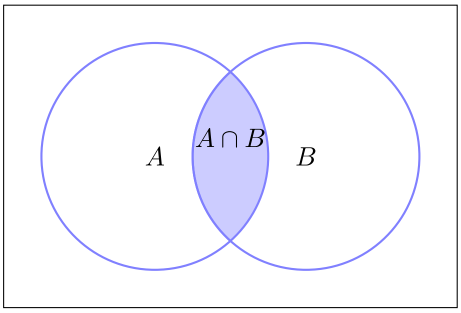
Diagramatic set theory

Venn Diagrams

Venn diagrams are used to illustrate sets. The bounding rectangle represents the universe U and the interiors of circles represent subsets of U. For larger numbers of sets the shapes of the sets are no longer circles (and the visualisation may be harder to interpret).

Example 20. Draw the Venn diagram for $A \cap B$ where A and B are sets.

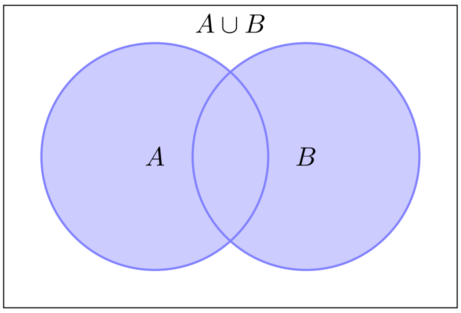
Proof. .



□

Example 21. Draw the Venn diagram for $A \cup B$, where A and B are sets.

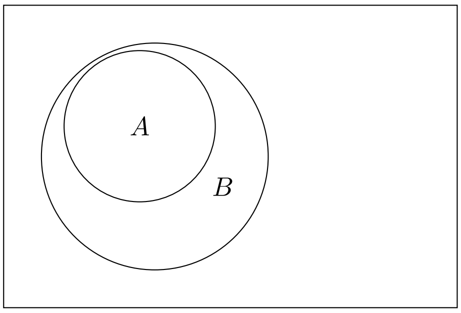
Proof. .



□

Example 22. Draw the Venn diagram for sets A and B with $A \subseteq B$.

Proof. .



□

Exercise 20. Draw the Venn diagram for the symmetric difference $A \Delta B$ of two sets A and B.

Exercise 21. Draw the Venn diagram for the difference $A \setminus B$ of two sets A and B.

Exercise 22. Identify the following regions in a Venn diagram for two sets A and B:

(a) $A \cup B^c$,

(b) $A^c \cap B$,

(c) $(A \cup B)^c$,

(d) $(A^c \cap B^c)^c$,

(e) $(A \cup B) \setminus (A \cap B)$,

(f) $A \cap (B \cup A^c)$.

Exercise 23. Draw the Venn diagram for the sets $A_1, \dots, A_n$ where only A_i, A_{i+1} have non-empty intersections. Draw the Venn diagram if in addition A_n, A_1 have non-empty intersection.

Exercise 24. By drawing a Venn diagram for finite sets A and B, show that $\#(A \cup B) = \#A + \#B - \#(A \cap B)$.

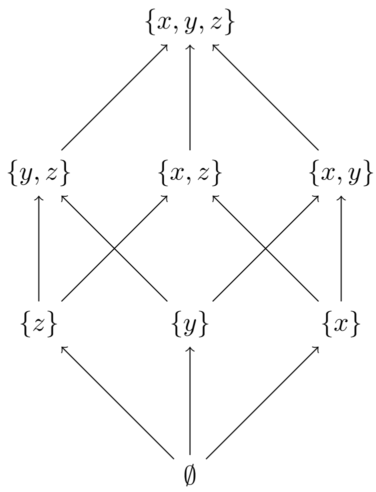
Exercise 25. By drawing a Venn diagram for finite sets A, B and C, show that $\#(A \cup B \cup C) = \#A + \#B + \#C - \#(A \cap B) - \#(B \cap C) - \#(C \cap A) + \#(A \cap B \cap C)$. (optional) Can you generalise for n sets? (*Hint:* Take a look at the principle of inclusion/exclusion.)

Hasse Diagrams

The Hasse diagrams is the graph visualisation of the $\subset$ relation but with the transitive links removed (see relations handout).

Example 23. Draw the Hasse Diagram for $\mathcal{P}(\{x, y, z\})$.

Proof. .



□

Exercise 26. Draw the Hasse diagrams for $\{D(i) : i \in \mathbb{N}, 1 \leq i \leq 24\}$, where $D(n)$ is the set of divisors for n.

Exercise 27. Is it possible for a Hasse diagram to contain a cycle?

Exercise 28. Do all Hasse diagrams have an element with no incoming edges? Is this element always unique? Prove or disprove by proving a counterexample.

Exercise 29. Do all Hasse diagrams have an element with no outgoing edges? Is this element always unique? Prove or disprove by proving a counterexample.

Past papers

COMPUTER SCIENCE TRIPOS Part IA – 2018 – Paper 2

9 Discrete Mathematics (MPF)

(b) Let U be a set and let $\mathcal{P}(U)$ denote its powerset. For $\mathcal{F} \subseteq \mathcal{P}(U)$, define $\mathcal{G} \subseteq \mathcal{P}(U)$ as $\{X \subseteq U \mid \forall S \in \mathcal{F}. S \subseteq X\}$

Prove that $\bigcup \mathcal{F} = \bigcap \mathcal{G}$. [4 marks]

COMPUTER SCIENCE TRIPOS Part IA – 2015 – Paper 2

7 Discrete Mathematics (MPF)

(*c*) Let U be a set and let $F : \mathbb{N} \times \mathbb{N} \to \mathcal{P}(U)$ be a function such that for all $i, i', j, j' \in \mathbb{N}$, if $i \leq i'$ and $j \leq j'$ then $F(i, j) \subseteq F(i', j')$ in $\mathcal{P}(U)$.

Prove that

$$\bigcup_{i\in\mathbb{N}} \left(\bigcup_{j\in\mathbb{N}} F(i, j)\right) = \bigcup_{k\in\mathbb{N}} F(k, k)$$

(Recall that $x \in \bigcup_{l\in L} X_l \iff \exists\, l \in L.\ x \in X_l$.) [6 marks]

COMPUTER SCIENCE TRIPOS Part IA – 2013 – Paper 1

3 Discrete Mathematics I (SS)

(*a*) Consider the following assertions about the sets A, B and C. Write them down in the language of predicate logic. Use only the constructions of predicate logic ($\forall$, $\exists$, $\neg$, $\Rightarrow$, $\wedge$, $\vee$) and the element-of symbol ($\in$). Do *not* use derived notions ($\cap$, $\cup$, $=$, etc.).

Example: "A is a subset of B" can be formalized as $\forall x.\, x \in A \implies x \in B$.

(*i*) The sets A and B are equal.

(*ii*) Every element of A is in the set B or the set C.

(*iii*) If A is disjoint from B then B and C overlap.

[6 marks]

COMPUTER SCIENCE TRIPOS Part IA – 2009 – Paper 1

3 Discrete Mathematics I (SS)

(*c*) For a set of sets A, write $\bigcup A$ for the set $\{x \mid \exists X \in A. x \in X\}$. For a non-empty set of sets A, write $\bigcap A$ for the set $\{x \mid \forall X \in A. x \in X\}$.

Suppose $A \subseteq \mathcal{P}(X)$ and $B \subseteq \mathcal{P}(X)$. Prove or give a counterexample for each of the following.

(*i*) If $\bigcup A$ and $\bigcup B$ are disjoint, then A and B are disjoint.

(*ii*) If A and B are disjoint then $\bigcup A$ and $\bigcup B$ are disjoint.

(*iii*) If A and B are non-empty and $\forall X \in A. \forall Y \in B. X \subseteq Y$ then $\bigcup A \subseteq \bigcap B$.

(*iv*) (not-in-the-exam) If A and B are non-empty and $\forall X \in A. \exists Y \in B. X \subseteq Y$ then $\bigcap A \subseteq \bigcup B$.

[9 marks]

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8 Discrete Mathematics (MPF)

Let Ω be a set. Write $\mathcal{P}(\Omega)$ for its powerset. Recall the definition of the intersection of $\mathcal{B} \subseteq \mathcal{P}(\Omega)$:

$$\bigcap_{B \in \mathcal{B}} B = \{x \in \Omega \mid \forall B \in \mathcal{B}.\ x \in B\}\ .$$

(a) Let $\mathcal{B} \subseteq \mathcal{P}(\Omega)$ and $\mathcal{C} \subseteq \mathcal{P}(\Omega)$.

(i) Prove that

$$(\bigcap_{B \in \mathcal{B}} B) \cup (\bigcap_{C \in \mathcal{C}} C) \ \subseteq \bigcap_{(B,C) \in \mathcal{B} \times \mathcal{C}} (B \cup C) \ .$$

[3 marks]

(ii) Prove that

$$\bigcap_{(B,C) \in \mathcal{B} \times \mathcal{C}} (B \cup C) \ \subseteq \ (\bigcap_{B \in \mathcal{B}} B) \cup (\bigcap_{C \in \mathcal{C}} C) \ .$$

[6 marks]

(b) Let $\mathcal{A} \subseteq \mathcal{P}(\Omega)$. Suppose that $\mathcal{A}$ is *intersection-closed* in the sense that

$$\text{if } \mathcal{B} \subseteq \mathcal{A}, \text{ then } \bigcap_{B \in \mathcal{B}} B \in \mathcal{A} \ .$$

Define

$$\mathcal{R} = \{(X, y) \in \mathcal{P}(\Omega) \times \Omega \mid \forall A \in \mathcal{A}.\ X \subseteq A \Rightarrow y \in A\} \ .$$

Let $C \subseteq \Omega$. Say C is $\mathcal{R}$-*closed* iff

$$\forall (X, y) \in \mathcal{R}.\ X \subseteq C \Rightarrow y \in C \ .$$

You are asked to show that the members of $\mathcal{A}$ are precisely the $\mathcal{R}$-closed subsets of Ω, in the following two stages:

(i) Show

$$\text{if } C \in \mathcal{A}, \text{ then } C \text{ is } \mathcal{R}\text{-closed} \ .$$

[2 marks]

(*ii*) Show

$$\text{if } C \text{ is } \mathcal{R}\text{-closed, then } C \in \mathcal{A} .$$

[Hint: Consider the set $\mathcal{B} = \{A \in \mathcal{A} \mid C \subseteq A\}$.] [9 marks]

Further problems

(optional) Problems on ZF

Here is a list of exercises that may help you better understand the ZF axioms. These are not part of the course.

Exercise 30. N. Wiener in 1914, used the following more complicated definition for ordered pairs $\langle a, b\rangle = \{\{\{a\}, \emptyset\}, \{\{b\}\}\}$. Show that this definition also satisfies $\langle a, b\rangle = \langle c, d\rangle \Rightarrow a = c \wedge b = d$.

Exercise 31. (Hamilton 4.2.8) Using (ZF5) and (ZF6) (and not (ZF3)) show that for any set x, there is a set which has x as it only element.

Exercise 32. (Hamilton 4.2.13) Given a set x, use the ZF axioms to deduce the existence of the set $\{\{a\} \mid a \in x\}$.

Exercise 33. (Hamilton 4.2.14) Deduce (ZF3) as a consequence of (ZF7) and (ZF8).

Exercise 34. (Hamilton 4.2.15) Consider the following weak version of (ZF3): Given sets x and y, there is a set S with $x \in S$ and $y \in S$. Call this (ZF3'). show that (ZF3') and (ZF6) imply (ZF3).

Exercise 35. (Hamilton 4.2.16) Write down weak versions for (ZF4) and (ZF5) and show that together with (ZF6), each implies the corresponding strong version.

Exercise 36. (Hamilton 4.2.17) Give an example of an infinite sequence $y_0, y_1, \ldots$ with $y_n \subseteq y_{n+1}$.

Exercise 37. (Hamilton 4.2.18) Can there exist sets x and y with $x \in y$ and $y \in x$? Is it possible for $x \in y, y \in z, z \in x$? Generalise.

Exercise 38. (Hamilton 4.2.19) Write down a statement which is equivalent to (ZF8), but which does not include any occurrence of the symbol $\emptyset$. Show that this alternative infinity axiom together with (ZF6) implies (ZF2).

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