

# Logic Exercises for Part IA Discrete Mathematics

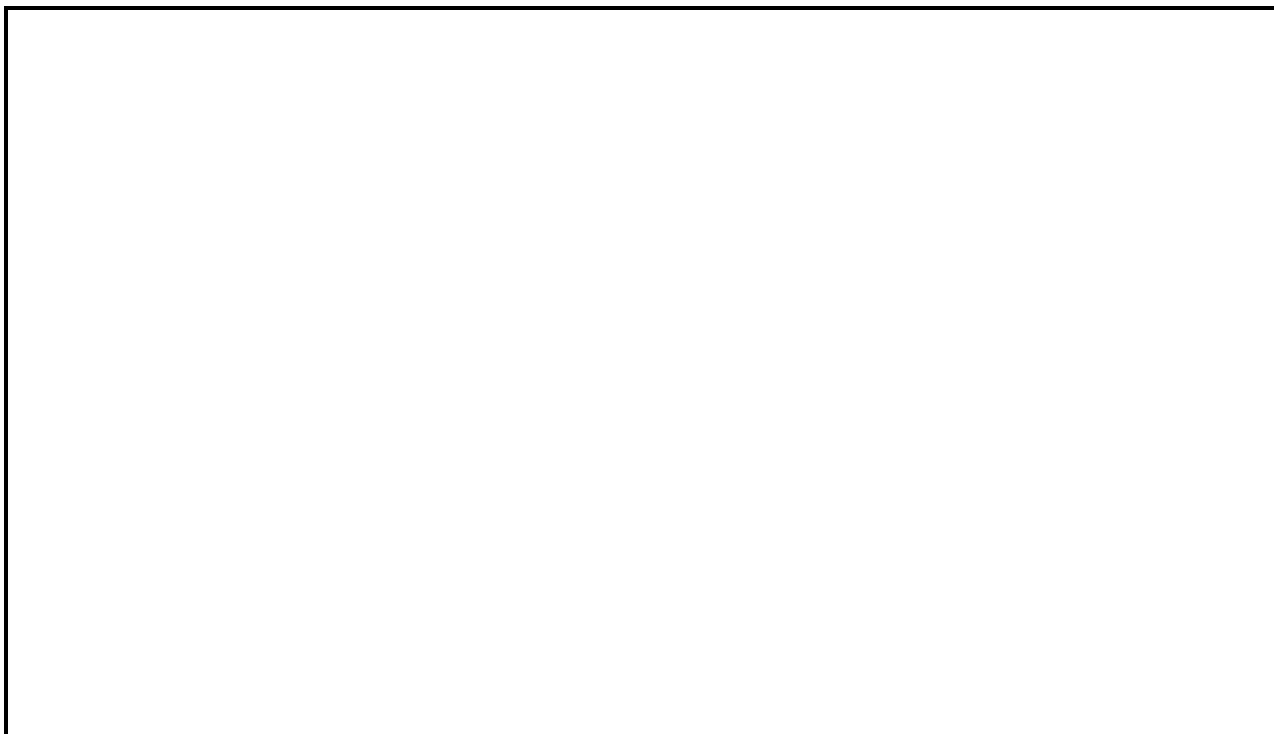
**Note** Most of these exercises are from the old format of the course. Recent proof exercises tend to involve proof by induction, so you will find these in the induction handout.

7 Discrete Mathematics (mpf23)

(a) Prove that, for all statements  $P$  and  $Q$ ,

$$(P \implies Q) \implies ((P \implies \neg Q) \implies \neg P)$$

[4 marks]



4 Discrete Mathematics I (SS)

- (a) Write down the introduction and elimination rules for the universal quantifier ( $\forall$ ), the existential quantifier ( $\exists$ ) and negation ( $\neg$ ) in structured proof. [6 marks]

- (b) Write down the introduction rule for implication ( $\implies$ ) in structured proof. [1 mark]

- (c) Write down a structured proof of the following sentence.

$$(\forall x. \neg P(x)) \implies \neg \exists x. P(x)$$

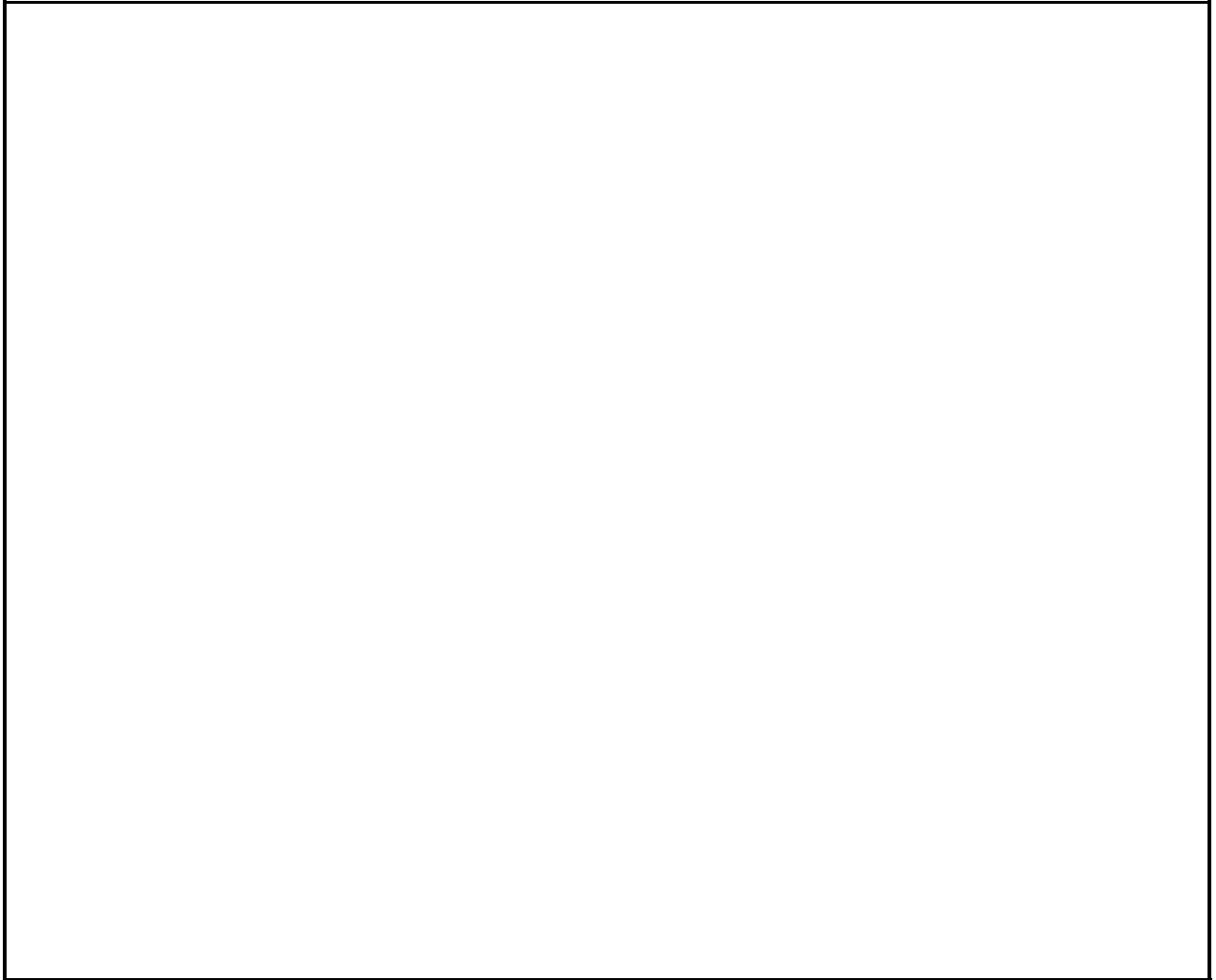
[5 marks]



(d) Write down a structured proof of the following sentence. Clearly state any proof rules that you use in addition to those included in part (a) and part (b).

$$(\neg\forall x. \neg P(x)) \implies \exists x. P(x)$$

[8 marks]



3 Discrete Mathematics I (SS)

(a) Which of the following formulas are tautologies? Explain what is meant by “tautology” and write down truth tables to justify your answers.

(i)  $p \Rightarrow q$

(ii)  $(p \Rightarrow q) \Rightarrow p$

(iii)  $((p \Rightarrow q) \Rightarrow p) \Rightarrow p$

[4 marks]

(b) Recall the following introduction and elimination rules for implication.

$\dots$   

$m.$ Assume $P$ $\dots$ $n.$ $Q$ from $\dots$ by $\dots$
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 $n + 1.$   $P \Rightarrow Q$  from  $m-n$ ,  
by  $\Rightarrow$ -introduction.

$\dots$   
 $l.$   $P \Rightarrow Q$  from  $\dots$  by  $\dots$   
 $\dots$   
 $m.$   $P$  from  $\dots$  by  $\dots$   
 $\dots$   
 $n.$   $Q$  from  $l, m$   
by  $\Rightarrow$ -elimination.

(i) Write down the elimination rules for negation and falsity.

[3 marks]

(ii) Using the four rules above, write down a structured proof of

$$\neg p \Rightarrow (p \Rightarrow q)$$

[4 marks]

(iii) Write down the principle of proof by contradiction.

[2 marks]

(iv) Using everything from part (b) so far, write down a structured proof of

$$((p \Rightarrow q) \Rightarrow p) \Rightarrow p$$

[7 marks]



4 Discrete Mathematics I (SS)

- (c) Write down the introduction and elimination rules for the universal quantifier in structured proof. [3 marks]

- (d) Recall the following introduction and elimination rules for implication.

$\frac{\begin{array}{l} \dots \\ m. \text{ Assume } P \\ \dots \\ n. Q \text{ from } \dots \text{ by } \dots \end{array}}{n + 1. P \Rightarrow Q \text{ from } m\text{--}n, \\ \text{by } \Rightarrow\text{-introduction.}}$	$\frac{\begin{array}{l} \dots \\ l. P \Rightarrow Q \text{ from } \dots \text{ by } \dots \\ \dots \\ m. P \text{ from } \dots \text{ by } \dots \\ \dots \\ n. Q \text{ from } l, m \end{array}}{\text{by } \Rightarrow\text{-elimination.}}$
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Write down a structured proof of the following statement.

$$(\forall a. P(a) \Rightarrow Q(a)) \Rightarrow ((\forall b. Q(b) \Rightarrow R(b)) \Rightarrow (\forall c. P(c) \Rightarrow R(c)))$$

[8 marks]





3 Discrete Mathematics I (SS)

This question is about structured proofs.

(a) Write down the introduction and elimination rules for implication and negation.

[4 marks]

(b) Using the rules from part (a), give a structured proof of

$$(P \Rightarrow Q) \Rightarrow ((\neg Q) \Rightarrow (\neg P))$$

[7 marks]

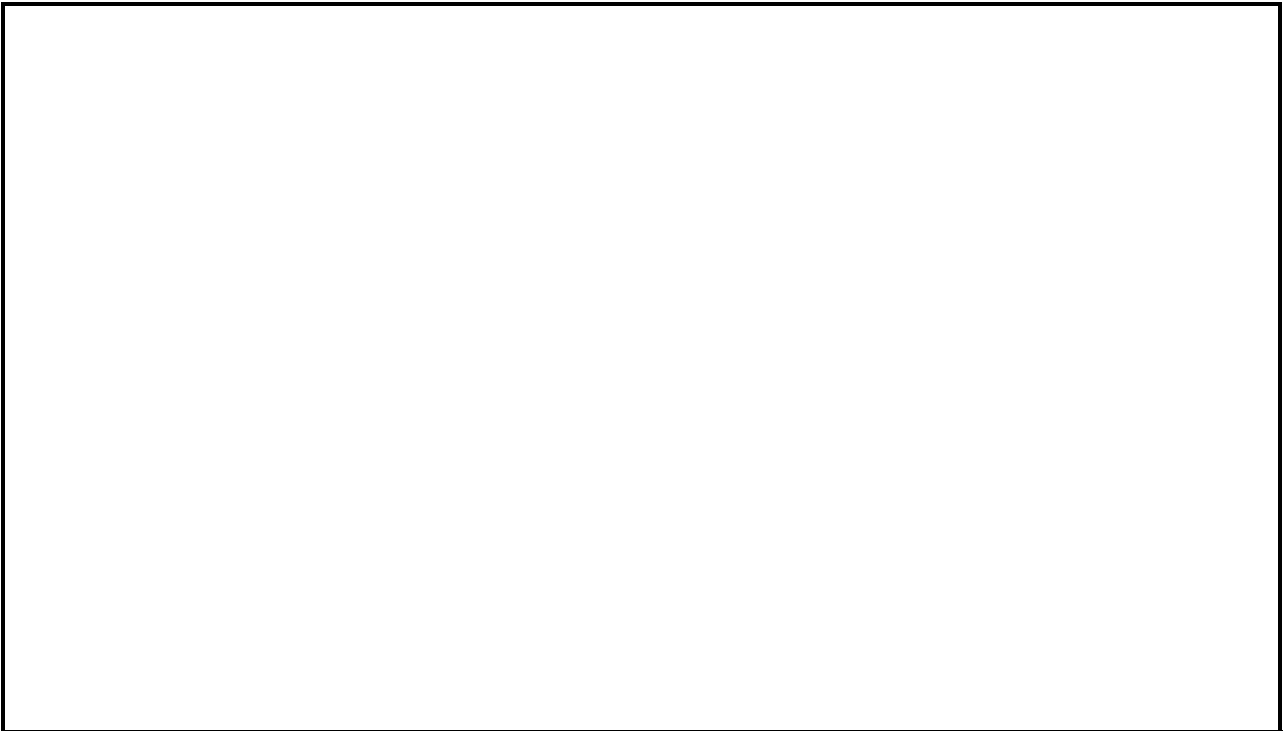
(c) Write down the rule for proof by contradiction.

[2 marks]



(d) Using the rules from parts (a) and (c), give a structured proof of

$$((\neg Q) \Rightarrow (\neg P)) \Rightarrow (P \Rightarrow Q) \quad [7 \text{ marks}]$$



4 Discrete Mathematics I (SS)

Let  $x, y, z$  range over individuals  $I$  and  $a, b$  range over societies  $S$ . Let  $M, F$  and  $T$  be atomic predicates as follows:

$M(x, a)$	$x$ is a member of society $a$
$F(a)$	society $a$ involves fighting
$T(x, y, a)$	$x$ talks to $y$ about $a$

(a) Formalise each of the following English statements and translate each of the following formulae into idiomatic English (natural English sentences).

(i)  $\forall x, y, a. T(x, y, a) \Rightarrow T(y, x, a)$

(ii) Nobody talks to themselves about anything.

(iii) There's at most one society involving fighting.

(iv) All societies have at least two members.

(v)  $\forall a. (\exists x, y. (M(x, a) \wedge M(y, a) \wedge x \neq y)) \Rightarrow$   
 $\exists x, y, b. M(x, a) \wedge M(y, a) \wedge x \neq y \wedge T(x, y, b) \wedge F(b)$

(vi)  $\forall x, y, a. T(x, y, a) \Rightarrow M(x, a)$

[12 marks]

- (b) Is it possible to satisfy  $(a)(i)-(a)(vi)$  simultaneously? Either give a concrete definition of two sets  $I$  and  $S$  and relations  $M$ ,  $F$ , and  $T$  for which  $(a)(i)-(a)(vi)$  are all true or prove that you can derive a contradiction from  $(a)(i)-(a)(vi)$ .

[4 marks]

- (c) Here are several attempts to formalise “Somebody talks about everything”. Explain what they actually mean, discussing whether or not each is a reasonable formalisation.

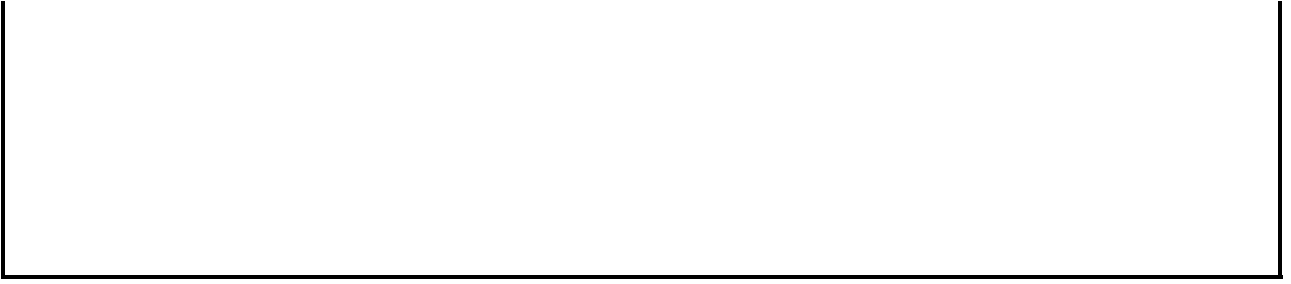
(i)  $\exists x.\forall a.\exists y.T(x, y, a)$

(ii)  $\exists x.\exists y.\forall a.T(x, y, a)$

(iii)  $\forall x.\forall a.\exists y.T(x, y, a)$

(iv)  $\exists y.\forall a.\forall x.T(x, y, a)$

[4 marks]



5 Discrete Mathematics I (SS)

- (a) State the structured-proof rules for implication introduction and disjunction elimination. [3 marks]

- (b) Give either a structured proof or a counterexample for each of the following.

(i)  $((P \Rightarrow Q) \vee (P \Rightarrow R)) \Rightarrow (P \Rightarrow (Q \vee R))$

(ii)  $((P \wedge Q) \Rightarrow R) \Rightarrow ((P \Rightarrow R) \wedge (Q \Rightarrow R))$

[8 marks]





6 Discrete Mathematics II (SS)

- (a) (i) Draw the truth tables to illustrate the truth values of  $A \Rightarrow B$  and  $A \Leftrightarrow B$  in terms of the truth values of A and B. [2 marks]

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- (ii) By considering their truth tables, establish the following equivalences of boolean propositions:

(A)  $A \Leftrightarrow (B \Leftrightarrow C) = (A \Leftrightarrow B) \Leftrightarrow C$ . [5 marks]

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(B)  $(F \Leftrightarrow B) = \neg B$ , where F is the proposition “false”. [2 marks]

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(C)  $\neg(B \Leftrightarrow C) = ((\neg B) \Leftrightarrow C)$ .

[2 marks]

(iii) By assigning suitable truth values to propositions B and C, explain why the equivalence (a)(ii)(C) above fails to hold if “ $\Leftrightarrow$ ” is replaced by “ $\Rightarrow$ ”. [3 marks]