Logic Exercises for Part IA Discrete Mathematics

Note Most of these exercises are from the old format of the course. Recent proof exercises tend to involve proof by induction, so you will find these in the induction handout.

7 Discrete Mathematics (mpf23)

(*a*) Prove that, for all statements P and Q,

$$(P \implies Q) \implies ((P \implies \neg Q) \implies \neg P)$$

[4 marks]

4 Discrete Mathematics I (SS)

(*a*) Write down the introduction and elimination rules for the universal quantifier ($\forall$), the existential quantifier ($\exists$) and negation ($\neg$) in structured proof. [6 marks]

(*b*) Write down the introduction rule for implication ($\Longrightarrow$) in structured proof. [1 mark]

(*c*) Write down a structured proof of the following sentence.

$$(\forall x.\, \neg P(x)) \implies \neg \exists x.\, P(x)$$

[5 marks]

(*d*) Write down a structured proof of the following sentence. Clearly state any proof rules that you use in addition to those included in part (*a*) and part (*b*).

$$(\neg\forall x.\, \neg P(x)) \implies \exists x.\, P(x)$$

[8 marks]

3 Discrete Mathematics I (SS)

(*a*) Which of the following formulas are tautologies? Explain what is meant by "tautology" and write down truth tables to justify your answers.

(*i*) $\mathrm{p} \Rightarrow \mathrm{q}$

(*ii*) $(\mathrm{p} \Rightarrow \mathrm{q}) \Rightarrow \mathrm{p}$

(*iii*) $((\mathrm{p} \Rightarrow \mathrm{q}) \Rightarrow \mathrm{p}) \Rightarrow \mathrm{p}$ [4 marks]

(*b*) Recall the following introduction and elimination rules for implication.

...
| m. Assume P
| ...
| n. Q from ... by ...
$n+1$. $P \Rightarrow Q$ from m–n, by $\Rightarrow$-introduction.

...
l. $P \Rightarrow Q$ from ... by ...
...
m. P from ... by ...
...
n. Q from l, m by $\Rightarrow$-elimination.

(*i*) Write down the elimination rules for negation and falsity. [3 marks]

(*ii*) Using the four rules above, write down a structured proof of

$$\neg p \Rightarrow (p \Rightarrow q)$$

[4 marks]

(*iii*) Write down the principle of proof by contradiction. [2 marks]

(*iv*) Using everything from part (*b*) so far, write down a structured proof of

$$((p \Rightarrow q) \Rightarrow p) \Rightarrow p$$

[7 marks]

4 Discrete Mathematics I (SS)

(c) Write down the introduction and elimination rules for the universal quantifier in structured proof. [3 marks]

(d) Recall the following introduction and elimination rules for implication.

...	...
m. Assume P	l. $P \Rightarrow Q$ from ... by ...
...	...
n. Q from ... by ...	m. P from ... by ...
$n+1$. $P \Rightarrow Q$ from m–n, by $\Rightarrow$-introduction.	...
	n. Q from l, m by $\Rightarrow$-elimination.

Write down a structured proof of the following statement.

$$\big(\forall a.\, P(a) \Rightarrow Q(a)\big) \Rightarrow \Big(\big(\forall b.\, Q(b) \Rightarrow R(b)\big) \Rightarrow \big(\forall c.\, P(c) \Rightarrow R(c)\big)\Big)$$

[8 marks]

3 Discrete Mathematics I (SS)

This question is about structured proofs.

(*a*) Write down the introduction and elimination rules for implication and negation. [4 marks]

(*b*) Using the rules from part (*a*), give a structured proof of

$$(P \Rightarrow Q) \Rightarrow ((\neg Q) \Rightarrow (\neg P))$$ [7 marks]

(*c*) Write down the rule for proof by contradiction. [2 marks]

(*d*) Using the rules from parts (*a*) and (*c*), give a structured proof of

$$((\neg Q) \Rightarrow (\neg P)) \Rightarrow (P \Rightarrow Q)$$

[7 marks]

4 Discrete Mathematics I (SS)

Let x, y, z range over individuals I and a, b range over societies S. Let M, F and T be atomic predicates as follows:

$M(x, a)$	x is a member of society a
$F(a)$	society a involves fighting
$T(x, y, a)$	x talks to y about a

(*a*) Formalise each of the following English statements and translate each of the following formulae into idiomatic English (natural English sentences).

(*i*) $\forall x, y, a. T(x, y, a) \Rightarrow T(y, x, a)$

(*ii*) Nobody talks to themselves about anything.

(*iii*) There's at most one society involving fighting.

(*iv*) All societies have at least two members.

(*v*) $\forall a. (\exists x, y. (M(x, a) \wedge M(y, a) \wedge x \neq y)) \Rightarrow$
$\exists x, y, b. M(x, a) \wedge M(y, a) \wedge x \neq y \wedge T(x, y, b) \wedge F(b)$

(*vi*) $\forall x, y, a. T(x, y, a) \Rightarrow M(x, a)$

[12 marks]

(b) Is it possible to satisfy (a)(i)–(a)(vi) simultaneously? Either give a concrete definition of two sets I and S and relations M, F, and T for which (a)(i)–(a)(vi) are all true or prove that you can derive a contradiction from (a)(i)–(a)(vi).

[4 marks]

(c) Here are several attempts to formalise "Somebody talks about everything". Explain what they actually mean, discussing whether or not each is a reasonable formalisation.

(i) $\exists x.\forall a.\exists y.T(x, y, a)$

(ii) $\exists x.\exists y.\forall a.T(x, y, a)$

(iii) $\forall x.\forall a.\exists y.T(x, y, a)$

(iv) $\exists y.\forall a.\forall x.T(x, y, a)$

[4 marks]

5 Discrete Mathematics I (SS)

(*a*) State the structured-proof rules for implication introduction and disjunction elimination. [3 marks]

(*b*) Give either a structured proof or a counterexample for each of the following.

(*i*) $((P \Rightarrow Q) \vee (P \Rightarrow R)) \Rightarrow (P \Rightarrow (Q \vee R))$

(*ii*) $((P \wedge Q) \Rightarrow R) \Rightarrow ((P \Rightarrow R) \wedge (Q \Rightarrow R))$

[8 marks]

6 Discrete Mathematics II (SS)

(*a*) (*i*) Draw the truth tables to illustrate the truth values of $\mathtt{A} \Rightarrow \mathtt{B}$ and $\mathtt{A} \Leftrightarrow \mathtt{B}$ in terms of the truth values of $\mathtt{A}$ and $\mathtt{B}$. [2 marks]

(*ii*) By considering their truth tables, establish the following equivalences of boolean propositions:

(A) $\mathtt{A} \Leftrightarrow (\mathtt{B} \Leftrightarrow \mathtt{C}) = (\mathtt{A} \Leftrightarrow \mathtt{B}) \Leftrightarrow \mathtt{C}$. [5 marks]

(B) $(\mathtt{F} \Leftrightarrow \mathtt{B}) = \neg\mathtt{B}$, where $\mathtt{F}$ is the proposition "false". [2 marks]

(C) $\neg(\mathtt{B} \Leftrightarrow \mathtt{C}) = ((\neg\mathtt{B}) \Leftrightarrow \mathtt{C})$. [2 marks]

(*iii*) By assigning suitable truth values to propositions B and C, explain why the equivalence $(a)(ii)$(C) above fails to hold if "⇔" is replaced by "⇒". [3 marks]