# Fibonacci Numbers for Part IA Discrete Mathematics

**Note:** This handout contains several exercises and related past papers to the *Fibonacci numbers*

The material in this handout appears in multiple books and other resources, so you will find a more complete treatment there. For instance, Chapter 14 of "Elementary Number Theory" by David M. Burton (most of the exercises come from here) and "Fibonacci Numbers" by N. Vorobiev. Also, several innovations regarding the Fibonacci numbers along with a problem list appear in the "Fibonacci Quarterly" (see this archive).

Fibonacci numbers appear in several places in CS, such as the analysis of algorithms (gcd algorithm (Part IA DM), Fibonacci Heaps (Part IA Algorithms)) and computation theory (crucial in the solution of Hilbert's tenth problem (related to Part IB Computation Theory)).

# Recursive formula

**Definition 1.** The Fibonacci sequence is defined by $F_0 = 0$, $F_1 = 1$, and for natural $n \geq 0$,

$$F_{n+2} := F_{n+1} + F_n.$$

# Closed-form solution

**Theorem 1.** Show that the $n$-th Fibonacci is given by

$$F_n = \frac{1}{\sqrt{5}} \cdot \left(\phi^n - (-\phi)^{-n}\right) = \frac{1}{\sqrt{5}} \cdot \left( \left( \frac{1+\sqrt{5}}{2} \right)^n - \left( \frac{1-\sqrt{5}}{2} \right)^n \right)$$

where $\phi = \frac{1+\sqrt{5}}{2}$ and $\psi = -\phi^{-1} = \frac{1-\sqrt{5}}{2}$. *Hint: use strong induction starting from $n = 2$ and show that $\phi^2 = 1 + \phi$ and $\phi^{-2} = 1 - \phi^{-1}$.*

**Note:** The term $\frac{1+\sqrt{5}}{2}$ is the golden ratio number $\phi$ and the number $\frac{1-\sqrt{5}}{2}$ is the (less-known) $\psi$.

**Note:** that $\phi = 1.61.. > 1$ and $|\psi| = 0.618.. < 1$, so as $n$ grows large the dominating term in the expression is $\frac{\phi^n}{\sqrt{5}}$. See the table below for the approximation of the $n$-th Fibonacci number using just $\phi^n/\sqrt{5}$.

| $F_3$ | $F_4$ | $F_5$ | $F_6$ | $F_7$ | $F_8$ | $F_9$ | $F_{10}$ | $F_{11}$ | $F_{12}$ | $F_{13}$ | $F_{14}$ | $F_{15}$ | $F_{16}$ | $F_{17}$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 2 | 3 | 5 | 8 | 13 | 21 | 34 | 55 | 89 | 144 | 233 | 377 | 610 | 987 | 1597 |
| 1.89 | 3.07 | 4.96 | 8.02 | 13.0 | 21.0 | 34.0 | 55.0 | 89.0 | 144. | 233. | 377. | 610. | 987. | 1600. |

*Why does this formula give integer numbers?* Well one reason is that it evaluates to the Fibonacci numbers, which are integers (as sum of integers from the recursive formula). OK, but this is not very illuminating. Where do the $\sqrt{5}$ go? Let's expand out the terms using the Binomial theorem,

$$\begin{aligned}
F_n &= \frac{1}{\sqrt{5}} \cdot \left( \left( \frac{1+\sqrt{5}}{2} \right)^n - \left( \frac{1-\sqrt{5}}{2} \right)^n \right) \\
&= \frac{1}{\sqrt{5}} \left( \frac{1}{2^n} \sum_{i=0}^{n} \binom{n}{i} (\sqrt{5})^i 1^{n-i} - \frac{1}{2^n} \sum_{i=0}^{n} \binom{n}{i} (-\sqrt{5})^i 1^{n-i} \right) \\
&= \frac{1}{\sqrt{5} \cdot 2^n} \left( \sum_{i=0}^{n} \binom{n}{i} (\sqrt{5})^i - \binom{n}{i} (-\sqrt{5})^i \right) \\
&= \frac{1}{\sqrt{5} \cdot 2^n} \left( \sum_{i=0}^{n} \binom{n}{i} \left( (\sqrt{5})^i - (-\sqrt{5})^i \right) \right)
\end{aligned}$$

Notice that for $i$ odd, i.e. $2k+1$ for $k \in \mathbb{Z}$, $(\sqrt{5})^{2k+1} + (\sqrt{5})^{2k+1} = \sqrt{5}((\sqrt{5})^{2k} + (\sqrt{5})^{2k}) = 2\sqrt{5} \cdot 5^k$. For $i$ even, we have $(\sqrt{5})^i - (\sqrt{5})^i = 0$. Hence, only the odd terms remain,

$$\begin{aligned} F_n &= \frac{1}{\sqrt{5} \cdot 2^{n-1}} \left( \binom{n}{1}\sqrt{5} + \binom{n}{3}\sqrt{5} \cdot 5 + \binom{n}{5}\sqrt{5} \cdot 5^2 + \ldots + + \binom{n}{2i+1}\sqrt{5} \cdot 5^i + \ldots \right) \\ &= \frac{1}{2^{n-1}} \left( \binom{n}{1} + \binom{n}{3} \cdot 5 + \binom{n}{5} \cdot 5^2 + \ldots + + \binom{n}{2i+1} 5^i + \ldots \right) \end{aligned}$$

So, the $\sqrt{5}$ values cancel out. The fact that $2^{n-1} \mid \binom{n}{1} + \binom{n}{3} \cdot 5 + \binom{n}{5} \cdot 5^2 + \ldots + + \binom{n}{2i+1} 5^i + \ldots$ will be proven in the Binomial handout.

# (optional) 2nd-order linear recurrence relations

*How did we come up with that formula?* The answer comes from a general method of solving linear recurrence relations. The method for solving a recurrence relation of the form $a_{n+2} = A \cdot a_{n+1} + B \cdot a_n$ (when $a_0$ and $a_1$ are known) is the following:

($a$) Replace $a_{n+2}$ with $x^2$, replace $a_{n+1}$ with $x$ and $a_n$ with 1. To get an equation of the form $x^2 = Ax + B$ (called the *characteristic equation*).

($b$) Solve the equation $x^2 = Ax + B$ or $x^2 - Ax - B = 0$ to find $x_1$ and $x_2$.

($c$) Based on $x_1$ and $x_2$, we can determine the form of the equation for $C_1$ and $C_2$ constants.

   ($i$) If $x_1 = x_2$ then the general solution has the form $a_n = C_1 x_1^n + C_2 x_1^n \cdot n$.

   ($ii$) If $x_1 \neq x_2$, then the general solution has the form $a_n = C_1 x_1^n + C_2 x_2^n$.

($d$) Find constants $C_1$, $C_2$ by plugging in $n = 0$, $n = 1$ and equating with the given values $a_0$ and $a_1$.

This technique generalises to higher order linear recurrence relations (see p.359 and forward in "Discrete Mathematics with Proof" (2nd edition) by Eric Gossett).

*How is it applied for the Fibonacci case?*

For the Fibonacci numbers $F_{n+2} = F_{n+1} + F_n$ and $F_0 = 0$ and $F_1 = 1$. Hence, the characteristic equation is $x^2 - x = 1 = 0$, which has two roots $x_1 = \frac{1+\sqrt{5}}{2}$ and $x_2 = \frac{1-\sqrt{5}}{2}$. So, the solution has the form

$$F_n = C_1 \cdot x_1^n + C_2 \cdot x_2^n = C_1 \cdot \left(\frac{1+\sqrt{5}}{2}\right)^n + C_2 \cdot \left(\frac{1-\sqrt{5}}{2}\right)^n,$$

which explains the presence of $\phi$ and $\psi$. For $n = 0$, we get $F_0 = 0 = C_1 + C_2$ and for $n = 1$ we get $C_1 \cdot \left(\frac{1-\sqrt{5}}{2}\right) + C_2 \cdot \left(\frac{1+\sqrt{5}}{2}\right) = 1$, so $C_1 = -C_2 = \frac{1}{\sqrt{5}}$. This gives,

$$F_n = \frac{1}{\sqrt{5}} \cdot \left( \left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n \right)$$

**Exercise 1.** Write a short program to compute the first ten terms for a linear recurrence relation of the form $a_{n+2} = A \cdot a_{n+1} + B \cdot a_n$. Find the closed-form solution for the following linear recurrence relations and verify that the terms match.

($a$) $a_{n+2} = -a_{n+1} + 2a_n$ with $a_0 = 5$ and $a_1 = 3$.

($b$) $a_{n+2} = -4a_{n+1} - 3a_n$ with $a_0 = 2$ and $a_1 = 4$.

($c$) $a_{n+2} = -6a_{n+1} - 9a_n$ with $a_0 = 6$ and $a_1 = 2$.

**Exercise 2.** (optional) What happens if the roots of the characteristic equations are not real? For example, take $a_{n+2} = a_{n+1} - a_n$ and $a_0 = 5$, $a_1 = 3$.

($a$) Write out the first 10 terms. What do you notice?

($b$) Determine the general formula for $a_n$.

($c$) Solve the formula using the above methodology. What answers do you get? (you should get a solution of the form $C \cdot \cos(\pi n/3 + \omega)$ or $C_1 \cos(\pi n/3) + C_2 \sin(\pi n/3)$)

*Why should these be solutions?*

In order to answer this questions, we need to prove some properties of the solutions of recurrence equations. If $b_n$ and $c_n$ solve the recurrence (i.e. $b_{n+2} = A \cdot b_{n+1} + B \cdot b_n$ and $c_{n+2} = A \cdot c_{n+1} + B \cdot c_n$), then so does $d_n = b_n + c_n$. To see this

$$Ad_{n+1} + Bd_n = A(b_{n+1} + c_{n+1}) + B(b_n + c_n) = (Ab_{n+1} + Bb_n) + (Ac_{n+1} + Bc_n) = b_{n+2} + c_{n+2} = d_{n+2}.$$

So $d_n$ is indeed also a solution.

Now let's look at solutions of the form $a_n = C \cdot r^n$ for $r \neq 0$, $C \neq 0$, i.e. for what values of $r$ this could be a solution.

$$a_{n+2} = A \cdot a_{n+1} + B \cdot a_n \Leftrightarrow C \cdot r^{n+2} = C \cdot Ar^{n+1} + CBr^n \Leftrightarrow r^n(r^2 - Ar - B) = 0$$

Hence, this is a solution iff $r = r_1$ or $r = r_2$ one of the roots of $r^2 - Ar - B = 0$. This shows that $b_n = C_1 r_1^n$ and $c_n = C_2 r^n$ are solutions. By the above, we have that $b_n + c_n$ is also a solution, which proves that $a_n = C_1 r_1^n + C_2 r_2^n$ is a solution.

When there is a unique root $r$, i.e. $r = -A/2$ and $B = -A^4/4$, we have that $a_n = Cr^n \cdot n$ is also a solution, since

$$a_{n+2} - Aa_{n+1} - Ba_n = Cr^{n+2}(n+2) - ACr^{n+1}(n+1) - BCr^n n = Cr^n((r^2 - Ar - B) + (r - A/2)) = 0$$

*Why are these solutions unique?*

To show that these solutions are unique we just need to show that for any $a_0$ and $a_1$ there exist coefficients $C_1$ and $C_2$, for which the formula agrees at $n = 0$ and $n = 1$. If the first two terms are determined by an inductive argument ($a_{n-1}$ and $a_{n-2}$ are determined, hence so is $a_n$) all terms are determined.

**(Distinct roots)** We need to show that the following system has a solution, where the unknowns are $C_1$ and $C_2$,

$$\begin{aligned} C_1 + C_2 &= a_0 \\ C_1 x_1 + C_2 x_2 &= a_1 \end{aligned}$$

There are various different ways that you can show that this has a solution for $x_1 \neq x_2$, the easiest being using checking that the coefficient determinant is not zero, i.e. $1 \cdot x_2 - 1 \cdot x_1 = x_2 - x_1 \neq 0$.

**(Equal roots)** We need to show that the following system has a solution, where the unknowns are $C_1$ and $C_2$,

$$\begin{aligned} C_1 + C_2 \cdot 0 &= a_0 \\ C_1 x_1 + C_2 x_1 &= a_1 \end{aligned}$$

Again, the coefficient determinant is not zero, i.e. $1 \cdot x_1 - 0 \cdot x_1 = x_1 \neq 0$, since $x_1 = x_2 = 0$ means that $A = B = 0$.

# (optional) Alternative derivation (under construction!)

*Can be skipped on first reading.* We already have shown how to solve linear recurrences. In case you still don't understand where solving the characteristic equation comes from, you may want to read the outline for another derivation.

Note that since $x_1$ and $x_2$ are the solutions to $x^2 - Ax - B = 0$, they satisfy $x_1 + x_2 = A$ and $x_1 x_2 = -B$. This allows to re-arrange the recurrence relation as,

$$a_{n+2} - (x_1 + x_2)a_{n+1} + x_1 x_2 a_n = 0 \Rightarrow \underbrace{a_{n+2} - x_1 a_{n+1}}_{b_{n+1}} = x_2 \underbrace{(a_{n+1} - x_1 a_n)}_{b_n}.$$

Now notice that the recurrence equation for $b_n$ is just a geometric progression, so $b_n = x_2^n b_0$, where $b_0 = a_1 - x_1 a_0$ which is just a constant. Hence, we have,

$$b_n = x_2^n b_0 = a_{n+1} - x_1 a_n \Rightarrow a_{n+1} = x_1 a_n + x_2^n b_0 = 0.$$

This does not look like any familiar sequence, but let's try to expand some of the terms.

$$\begin{aligned}
a_{n+1} &= x_1 a_n + x_2^n b_0 = x_1(x_1 a_{n-1} + x_2^{n-1} b_0) + x_2^n b_0 \\
&= x_1^2 a_{n-1} + x_1 x_2^{n-1} b_0 + x_2^n b_0 \\
&= x_1^2(x_1 a_{n-2} + x_2^{n-2} b_0) + x_1 x_2^{n-1} b_0 + x_2^n b_0 \\
&= x_1^3 a_{n-2} + x_1^2 x_2^{n-2} b_0 + x_1 x_2^{n-1} b_0 + x_2^n b_0 \\
&= x_1^{n+1} a_0 + b_0 \sum_{i=0}^{n} x_1^i x_2^{n-i} \\
&= x_1^{n+1} a_0 + b_0 x_2^n \sum_{i=0}^{n} \left(\frac{x_1}{x_2}\right)^i
\end{aligned}$$

Now we have to take two cases, if $x_1 = x_2$, then this gives $a_{n+1} = x^{n+1} a_0 + n b_0 x_1^n$ (which is exactly of the form that we proved above).

For $x_1 \neq x_2$, this gives $a_{n+1} = x_1^{n+1} a_0 + b_0 x_2^n \cdot \frac{\left(\frac{x_1}{x_2}\right)^{n+1} - 1}{\left(\frac{x_1}{x_2}\right) - 1} = C_1 x_1^{n+1} + C_2 x_2^{n+1}$ (Why?)

# Proving identities

In this section, we will investigate some techniques for proving properties of the Fibonacci numbers. These are not the only techniques and they are not hard rules.

The first method is by **expanding out** terms.
For example, we can expand $F_{n+4}$ into $F_{n+3} + F_{n+2}$ and then perhaps expand $F_{n+3}$, so we get $F_{n+4} = (F_{n+2} + F_{n+1}) + F_{n+2} = 2F_{n+2} + F_{n+1}$. This is also a way to obtain novel identities.
Note that the term $F_{n+4}$ appears in two other recurrence formulas, namely in $F_{n+5} = F_{n+4} + F_{n+3}$ and $F_{n+6} = F_{n+5} + F_{n+4}$. Hence, we can also use these expressions $F_{n+4} = F_{n+5} - F_{n+3} = F_{n+6} - F_{n+5}$.
Actually this first calculation, brings us to the first example,

**Example 1.** Show that for natural $n$,

$$F_{n+2} = \frac{F_{n+3} + F_n}{2}$$

*Proof.*We have two options from where to start. We could start from the LHS or the RHS. Let's start from the RHS. *Which terms should we expand?* $F_{n+3}$ does not appear in the LHS and will also introduce a $F_{n+2}$, so it reasonable to start from there.

$$\frac{F_{n+3} + F_n}{2} = \frac{(F_{n+2} + F_{n+1}) + F_n}{2}$$

Then we notice that $F_{n+1} + F_n$ is just $F_{n+2}$ so,

$$\frac{F_{n+2} + (F_{n+1} + F_n)}{2} = \frac{F_{n+2} + F_{n+2}}{2} = F_{n+2}$$

*What if we had started from $F_n$?* We have three options for what to replace $F_n$ with. Two of the options $F_n = F_{n-1} + F_{n-2}$ and $F_n = F_{n+1} - F_{n-1}$ include terms with smaller index, so they seem to make things worse. So, we choose the third option $F_n = F_{n+2} - F_{n+1}$,

$$\frac{F_{n+3} + F_{n+2} - F_{n+1}}{2} = \frac{(F_{n+3} - F_{n+1}) + F_{n+2}}{2} = \frac{(F_{n+2} + F_{n+2})}{2} = F_{n+2}$$

*What if we started from the LHS?* This is basically how we derived this identity in the first place, we replace $F_{n+2}$ once using $F_{n+3} - F_{n+1}$ and once using $F_{n+2} = F_{n+1} + F_n$.

$$2F_{n+2} = F_{n+2} + F_{n+2} = (F_{n+3} - F_{n+1}) + (F_{n+1} + F_n) = F_{n+3} + F_n \Rightarrow F_{n+2} = \frac{F_{n+3} + F_n}{2}$$

□

**Example 2.** Derive the identity for $n \geq 1$.

$$F_{n+3} = 3F_{n+1} - F_{n-1}$$

*Proof.* Here, again we have the choice to start from the LHS or the RHS. Let's start with the LHS, since expanding $F_{n+3}$ seems to help us,

$$F_{n+3} = F_{n+2} + F_{n+1}$$

Now, we expand $F_{n+2}$ (the one with largest index), so

$$F_{n+2} + F_{n+1} = F_{n+1} + F_n + F_{n+1} = 2F_{n+1} + F_n$$

Now, we are missing $F_{n+1}$ and a $-F_n$, so it is best to expand $F_n = F_{n+1} - F_{n-1}$.

$$2F_{n+1} + F_n = 2F_{n+1} + (F_{n+1} - F_{n-1}) = 3F_{n+1} - F_{n-1}$$

which gives us the desired identity.
*What if we start from the RHS?* We want to replace $F_{n-1}$ and $F_{n+1}$ with larger indices Fibonacci numbers, so let's start with the smallest one $F_{n-1} = F_{n+1} - F_n$,

$$3F_{n+1} - F_{n-1} = 3F_{n+1} - (F_{n+1} - F_n) = 2F_{n+1} + F_n$$

Again, we replace the smallest one, so

$$2F_{n+1} + F_n = 2F_{n+1} + (F_{n+2} - F_{n+1}) = F_{n+1} + F_{n+2}$$

where we recognise the last expression as $F_{n+3}$. □

Sometimes you need to make progress on both sides.

**Example 3.** Show that for $n \geq 2$,

$$F_{n+1}^2 - 4F_nF_{n-1} = F_{n-2}^2$$

*Proof.* Looking at the RHS, we know that if we expand $F_{n-2} = F_n - F_{n-1}$, then we get an expression that has the terms on the RHS, so intuitively it should help.

$$F_{n-2}^2 = (F_n - F_{n-1})^2 = F_n^2 - 2F_nF_{n-1} + F_{n-1}^2$$

It does not seem that we have a clear best choice, let's expand work on the LHS and expand $F_{n+1}$,

$$(F_n + F_{n-1})^2 - 4F_nF_{n-1} = F_n^2 + F_{n-1}^2 + 2F_nF_{n-1} - 4F_nF_{n-1} = F_n^2 - 2F_nF_{n-1} + F_{n-1}^2$$

□

**Note:** Nothing guarantees that the terms will be separated between LHS and RHS in a reasonable way. Sometimes it might make sense to bring everything on the same side.

**Example 4.** For $n \geq 2$, show that

$$F_n^2 - F_{n-1}^2 = F_{n+1}F_{n-2}.$$

*Proof.* Starting by factorising the LHS,

$$F_n^2 - F_{n-1}^2 = (F_n - F_{n-1})(F_n + F_{n-1}) = F_{n-2} \cdot F_{n+1}.$$

□

**Exercise 3.** Show that for $a_n = F_n^2 + F_{n-1}^2$, $b_n = 2F_nF_{n-1}$ and $c_n = F_{n-2}F_{n+1}$,

$$a_n^2 = b_n^2 + c_n^2.$$

**Exercise 4.** Show that for $a_n = F_{n+1}^2 + F_{n+2}^2$, $b_n = 2F_{n+1}F_{n+2}$ and $c_n = F_nF_{n+3}$,

$$a_n^2 = b_n^2 + c_n^2.$$

The next method is the well-known method of **proof by induction**, which also usually requires the expansion technique, especially in the induction step.

**Example 5.** Show that

$$F_0 + F_1 + \ldots + F_n = \sum_{i=1}^{n} F_i = F_{n+2} - 1$$

*Proof.* We will prove this by induction.
**(Base case)** For $F_2 - 1 = 1 - 1 = 0 = F_0$.
**(Induction step)** Assume true for $n = k$,

$$\sum_{i=1}^{k+1} F_i = \sum_{i=1}^{k} F_i + F_{k+1} = F_{k+2} - 1 + F_{k+1} = F_{k+2} + F_{k+1} - 1 = F_{k+3} - 1$$

So it is true for $n = k + 1$ and hence by the principle of mathematical induction it holds for all natural $n$. □

An alternative approach is to write down:

$$\begin{aligned} F_0 &= F_2 - F_1 \\ F_1 &= F_3 - F_2 \\ F_2 &= F_4 - F_3 \\ &\vdots \\ F_n &= F_{n+2} - F_{n+1} \end{aligned}$$

By summing up these equations many of the terms cancel out,

$$\begin{aligned}
F_0 &= \cancel{F_2} - F_1 \\
F_1 &= \cancel{F_3} - \cancel{F_2} \\
F_2 &= \cancel{F_4} - \cancel{F_3} \\
&\vdots \\
F_n &= F_{n+2} - \cancel{F_{n+1}}
\end{aligned}$$

Hence, on the LHS we have $\sum_{i=1}^{n} F_i$ and on the RHS we have $F_{n+2} - F_1 = F_{n+2} - 1$.

**Example 6** (Cassini's Identity)**.** For any $n \geq 1$,

$$F_n^2 = F_{n+1}F_{n-1} + (-1)^{n-1}$$

*Proof.* **Note:** It is clear that we cannot just use expansion in this problem because of the term $(-1)^{n-1}$. The expansion process will not generate any constant term.

(Induction step) Assume true for $n = k$, i.e. $F_k^2 = F_{k+1}F_{k-1} + (-1)^{k-1}$. Now consider $n = k + 1$, we start with the RHS,

$F_{k+2}F_k + (-1)^k = (F_{k+1} + F_k)F_k + (-1)^k = F_{k+1}F_k + F_k^2 + (-1)^k = F_{k+1}F_k + F_{k+1}F_{k-1} + (-1)^{k-1} + (-1)^k$

Notice that $(-1)^{k-1}$ and $(-1)^k$ are opposite, so they cancel out. Hence,

$$F_{k+1}F_k + F_{k+1}F_{k-1} = F_{k+1}(F_k + F_{k-1}) = F_{k+1} \cdot F_{k+1} = F_{k+1}^2.$$

□

**Example 7.** Show for $n \geq 2$, that

$$F_{n+1}F_{n-1} - F_{n+2}F_{n-2} = 2(-1)^n.$$

**Example 8.** Prove that for all natural numbers $k$ and $n$,

$$F_{n+k+1} = F_{k+1}F_{n+1} + F_kF_n.$$

*Proof.*Before we begin let's examine our options. Note that we are asked to prove a statement of the form $\forall n \in \mathbb{N}.\forall k \in \mathbb{N}.P(n,k)$. The principle of induction allows us to do one quantifier at a time, i.e. prove by induction on $n$ that $\forall k \in \mathbb{N}.P(n,k)$ and each statement for every $n$ will be proven by induction on $k$. So, let's begin.

**(Base case)** For $n = 0$, we need to show $\forall k \in \mathbb{N}.F_{0+k+1} = F_{k+1}F_{0+1} + F_kF_0$. We note that this identity is solvable using expansion, so there is no need for the nested induction,

$$F_{k+1}F_{0+1} + F_kF_0 = F_{k+1} \cdot 1 + F_k \cdot 0 = F_{k+1}.$$

**(Induction step)** Assume it is true for $n = \ell$, i.e. $\forall k \in \mathbb{N}.F_{\ell+k+1} = F_{k+1}F_{\ell+1} + F_kF_\ell$. We need to show $\forall k \in \mathbb{N}.F_{(\ell+1)+k+1} = F_{k+1}F_{(\ell+1)+1} + F_kF_{\ell+1}$.

So let $k$ be an arbitrary natural number,

$$F_{(\ell+1)+k+1} = F_{\ell+k+2} = F_{\ell+k+1} + F_{\ell+k} = F_{\ell+k+1} + F_{\ell+(k-1)+1}$$

By the induction hypothesis for $k$ and $k-1$ (note this is not strong induction),

$$\begin{aligned}
F_{\ell+k+1} + F_{\ell+(k-1)+1} &= F_{k+1}F_{\ell+1} + F_kF_\ell + F_kF_{\ell+1} + F_{k-1}F_\ell \\
&= F_{k+1}F_{\ell+1} + F_kF_\ell + F_{k-1}F_\ell + F_kF_{\ell+1} \\
&= F_{k+1}F_{\ell+1} + (F_k + F_{k-1})F_\ell + F_kF_{\ell+1} \\
&= F_{k+1}F_{\ell+1} + F_{k+1}F_\ell + F_kF_{\ell+1} \\
&= F_{k+1}(F_{\ell+1} + F_\ell) + F_kF_{\ell+1} \\
&= F_{k+1}F_{\ell+2} + F_kF_{\ell+1}
\end{aligned}$$

Hence, by the principle of ... .

**Note:** Below we give a proof of this formula using the closed-form solution and one using the matrix representation. □

**Exercise 5.** Show that the sum of the first $n$ Fibonacci numbers with odd indices is given by the formula,

$$F_1 + F_3 + F_5 + \ldots + F_{2n-1} = F_{2n}.$$

**Exercise 6.** Show that the sum of the first $n$ Fibonacci numbers with even indices is given by the formula,

$$F_0 + F_2 + F_4 + \ldots + F_{2n} = F_{2n+1} - 1.$$

**Exercise 7.** Show that the alternating sum of the first $n \geq 1$ Fibonacci numbers is given by the formula,

$$(-1)F_0 + F_1 + (-1)F_2 + F_3 + (-1)F_4 + \ldots + (-1)^{n+1}F_n = 1 + (-1)^{n+1}F_{n-1}$$

*Try also solving this problem using Exercise 6 and Exercise 5.*

Another method to prove an identity is to make use of an identity that you have already proven. This can save a lot of effort.

**Example 9.** Show that for $n \geq 1$, $F_nF_{n-1} = F_n^2 - F_{n-1}^2 + (-1)^n$.

*Proof.*We are going to use Cassini's identity, i.e. $F_n^2 = F_{n+1}F_{n-1} + (-1)^{n-1}$. Let's put the common terms on one side, i.e. Cassini's identity gives $F_n^2 + (-1)^n = F_{n+1}F_{n-1}$ and we want to prove that

$F_n^2 + (-1)^n = F_n F_{n-1} - F_{n-1}^2$.

$$F_n^2 + (-1)^n = F_{n+1}F_{n-1} = (F_n + F_{n-1})F_{n-1} = F_nF_{n-1} + F_{n-1}^2.$$

□

**Exercise 8.** Use Cassini's identity to show that for $n \geq 1$:

$(a)$ $F_{2n-1} = F_n^2 + F_{n-1}^2$

$(b)$ $F_{2n} = F_{n+1}^2 - F_{n-1}^2$

**Exercise 9.** Using the results from the previous exercise, show that

$(a)$ For $n \geq 2$, $F_{n+1}^2 + F_{n-2}^2 = 2F_{2n-1}$.

$(b)$ For $n \geq 1$, $F_{n+2}^2 + F_{n-1}^2 = 2(F_n^2 + F_{n+1}^2)$.

**Example 10.** Show that for $n \geq 2$,

$$F_n^2 - F_{n+2}F_{n-2} = (-1)^n.$$

*Proof.*

$$\begin{aligned}
F_n^2 - F_{n+2}F_{n-2} &= F_n^2 - (F_{n+1} + F_n)F_{n-2} \\
&= F_n(F_n - F_{n-2}) - F_{n+1}F_{n-2} \\
&= F_nF_{n-1} - (F_n + F_{n-1})F_{n-2} \\
&= F_{n-1}(F_n - F_{n-2}) - F_nF_{n-2} \\
&= F_{n-1}F_{n-1} - F_nF_{n-2} \\
&= F_{n-1}^2 - F_nF_{n-2} \\
&= (-1)^{n-2} = (-1)^n
\end{aligned}$$

where the last step follows from Cassini's identity for $n - 1$. □

**Exercise 10.** Show that for $n \geq 2$,

$$F_{n+1}F_{n-1} - F_{n+2}F_{n-2} = 2(-1)^n$$

*Hint: Start by expanding $F_{n+2}$ and $F_{n-2}$.*

**Exercise 11.** $(a)$ Show that for $n \geq 2$, $F_n^2 = F_nF_{n+1} - F_nF_{n-1}$.

$(b)$ Show that the sum of the squares of the first $n$ Fibonacci is given by

$$F_0^2 + F_1^2 + F_2^2 + \ldots + F_n^2 = \sum_{i=0}^{n} F_i^2 = F_nF_{n+1}$$

$(c)$ Show that for $n \geq 3$,

$$F_{n+1}^2 = F_n^2 + 3F_{n-1}^2 + 2(F_{n-2}^2 + F_{n-3}^2 + \ldots + F_2^2 + F_1^2)$$

Some of the identities can also be proven using the closed form solution. Usually this leads to long calculations and it requires some properties for $\phi$ and $\psi$, the main one being that they both satisfy $x^2 = x + 1$ and that $\psi = -\phi$ (*Why?*).

**Example 11.** For all natural numbers $n$ and $k$,

$$F_{n+k+1} = F_{k+1}F_{n+1} + F_kF_n.$$

*Proof.*

$$\begin{aligned}
&F_{k+1}F_{n+1} + F_kF_n \\
&\quad= \frac{1}{5} \cdot (\phi^{k+1} - (-\phi)^{-(k+1)})(\phi^{n+1} - (-\phi)^{-(n+1)}) + \frac{1}{5} \cdot (\phi^k - (-\phi)^{-k})(\phi^n - (-\phi)^{-n}) \\
&\quad= \frac{1}{5} \cdot (\phi^{k+n+2} + \cancel{(-1)^n\phi^{k-n}} + \cancel{(-1)^k\phi^{n-k}} + (-1)^{-(n+k+2)}\phi^{-(n+k+2)} \\
&\qquad + \phi^{k+n} + \cancel{(-1)^{k+1}\phi^{n-k}} + \cancel{(-1)^{n+1}\phi^{k-n}} + (-1)^{-(n+k)}\phi^{-(n+k)}) \\
&\quad= \frac{1}{5} \cdot (\phi^{k+n+2} + \phi^{k+n} + (-1)^{-(n+k+2)}\phi^{-(n+k+2)} + (-1)^{-(n+k)}\phi^{-(k+n)}) \\
&\quad= \frac{1}{5} \cdot (\phi^{k+n}(\phi^2 + 1) + (-1)^{-(n+k)}\phi^{-(n+k+2)}(1 + \phi^2)) \\
&\quad= \frac{1 + \phi^2}{5} \cdot (\phi^{k+n} + (-1)^{-(n+k)}\phi^{-(n+k+2)})
\end{aligned}$$

Now using the properties of $\phi$, $\frac{1}{\sqrt{5}} \cdot (1+\phi^2) = \frac{1}{\sqrt{5}} \cdot (1+(1+\phi)) = \frac{1}{\sqrt{5}} \cdot (2+\frac{1+\sqrt{5}}{2}) = \frac{1}{\sqrt{5}} \cdot \frac{5+\sqrt{5}}{2} = \frac{1+\sqrt{5}}{2} = \phi$. Hence,

$$\begin{aligned}
F_{k+1} &= \frac{\phi}{\sqrt{5}} \cdot (\phi^{k+n} + (-1)^{-(n+k)}\phi^{-(n+k+2)}) \\
&= \frac{1}{\sqrt{5}} \cdot (\phi^{k+n+1} + (-1)^{n+k}\phi^{-(n+k+1)}) \\
&= \frac{1}{\sqrt{5}} \cdot (\phi^{k+n+1} - (-1)^{n+k+1}\phi^{-(n+k+1)}) \\
&= F_{n+k+1}
\end{aligned}$$

□

# Divisibility properties

For computer science, Fibonacci numbers are particularly interesting because they provide a worst-case input for the Euclidean gcd algorithm. We will demonstrate why this is the case and also look at some other interesting divisibility properties of Fibonacci numbers. The main identity that we will be using is

$$F_{n+m+1} = F_{m+1}F_{n+1} + F_mF_n,$$

for which we provide three proofs in this document (induction, closed-form solution and matrix solution).

**Lemma 1.** For naturals $n \geq 1$, $\gcd(F_n, F_{n+1}) = 1$.

*Proof.* **(Using gcd properties)** We will prove this by induction. For $n = 1$, $\gcd(F_1, F_2) = \gcd(1, 2) = 1$. Assume true for $n = k$, so $\gcd(F_k, F_{k+1}) = 1$. Then for $n = k + 1$,

$$\gcd(F_{k+1}, F_{k+2}) = \gcd(F_{k+1}, F_k + F_{k+1}) = \gcd(F_{k+1}, F_k + F_{k+1} - F_{k+1}) = \gcd(F_{k+1}, F_k) = 1.$$

Hence, by the principle of ...

**(Using minimality)** We will use a technique of minimality. Assume that $n$ is the smallest natural such that $\gcd(F_n, F_{n+1}) = d > 1$. It has to be that $n > 1$ since $\gcd(F_1, F_2) = 1$. So $d \mid F_n$ and $d \mid F_{n+1}$. Hence, $d \mid F_{n+1} - F_n \Rightarrow d \mid F_{n-1}$. Hence, $\gcd(F_n, F_{n-1}) \geq d > 1$, but $n - 1 < n$, which contradicts that $n$ is the smallest such natural so it cannot exist. ☐

**Theorem 2.** Computing $\text{gcd0}(F_{n+1}, F_n)$ for any natural $n \geq 1$, takes $\Theta(n)$ steps.

*Proof.* We know that the Fibonacci numbers are strictly increasing for $n \geq 2$, so $F_n > F_{n-1}$. By definition, $F_{n+1} = F_n \cdot 1 + F_{n-1}$. By the uniqueness of the remainder (see Division Theorem) $F_{n-1} = \text{rem}(F_{n+1}, F_n)$. Hence, the next step of the algorithm gives $\text{gcd0}(F_n, \text{rem}(F_{n+1}, F_n)) = \text{gcd0}(F_n, F_{n-1})$. Hence, in each step the indices decrease by 1, so it will take $n - 1$ steps to reach $\texttt{gcd0}(F_2, F_1)$, where the algorithm terminates. ☐

**Note 1:** We could make this a formal induction argument, but sometimes in the analysis of algorithms, it is simpler to reason in this manner (though more error-prone).

**Note 2:** We proved in the second section that $F_n \approx \frac{\phi^n}{\sqrt{5}}$, hence $\log(F_n) = n \log(\phi) - \frac{1}{2}\log(5) = \Theta(n)$. This means that computing $\texttt{gcd0}(F_n, F_{n+1})$ takes $\Theta(n) = \Theta(\log F_n)$ steps, which establishes a lower bound on the time complexity of the `gcd0` which matches the upper bound (see the gcd handout).

**Lemma 2.** For naturals $n \geq 1$ and $m$, $F_{nm}$ is divisible by $F_n$.

*Proof.* We will prove this by induction over $m$, i.e. $\forall m \in \mathbb{N}. \forall n \in \mathbb{N}. n \geq 1 \Rightarrow F_n \mid F_{nm}$.

**(Base case)** For $m = 0$, $F_{mn} = F_0 = 0$, which is divisible by any natural, so by $F_n$ as well.

**(Induction step)** Assume true for $m = k$, i.e. $\forall n \in \mathbb{N}. n \geq 1 \Rightarrow F_n \mid F_{nk}$. Our goal is to prove that $\forall n \in \mathbb{N}. n \geq 1 \Rightarrow F_n \mid F_{n(k+1)}$. Let $n$ be arbitrary and assume $n \geq 1$. By the induction hypothesis, we know that $F_n \mid F_{nk}$ and we need to show that $F_n \mid F_{n(k+1)}$. So, we need to find the right parameters to use in the identity

$$F_{a+b+1} = F_{a+1}F_{b+1} + F_a F_b.$$

A reasonable aim is to have the LHS be $F_{n(k+1)}$. So we need $a + b = n(k + 1)$. Now we want to break the sum into $a$ and $b$ such that $nk$ appears. We can do this as $n(k + 1) = nk + n = (nk - 1) + n + 1$. One such choice is $a = nk - 1$ (but of course $b = nk - 1$ also works).

$$F_{nk+(n-1)+1} = F_{nk+1}F_{(n-1)+1} + F_{nk}F_{n-1} = F_{nk+1}F_n + F_{nk}F_{n-1}$$

By the induction hypothesis $F_n \mid F_{nk}$ so $F_n \mid F_{nk}F_{n-1}$. Also, $F_n \mid F_{nk+1}F_n$ so $F_n$ divides the sum $F_{nk+1}F_n + F_{nk}F_{n-1} = F_{n(k+1)}$. ☐

**Lemma 3.** If $m = qn + r$ for $r > 0$, then $\gcd(F_m, F_n) = \gcd(F_r, F_n)$.

*Proof.* Assume $m = qn + r$. Then using the identity $F_{a+b+1} = F_{a+1}F_{b+1} + F_a F_b$. We want $a + b + 1 = m$ and a reasonable splitting is $a = qn$ and $b = r - 1$ (The splitting $a = qn - 1$ and $b = r$ also works). Then,

$$F_m = F_{qn+1}F_{(r-1)+1} + F_{qn}F_{r-1} = F_{qn+1}F_r + F_{qn}F_{r-1}.$$

By the previous Lemmas we know that $F_n \mid F_{qn}$ and $\gcd(F_{qn}, F_{qn+1}) = 1$. Hence, by properties of the gcd,

$$\gcd(F_m, F_n) = \gcd(F_{qn+1}F_r + F_{qn}F_{r-1}, F_n) = \gcd(F_{qn+1}F_r, F_n) = \gcd(F_r, F_n).$$

□

**Lemma 4.** For any natural $n$ and $m$,

$$\gcd(F_n, F_m) = F_{\gcd(n,m)}.$$

*Proof.*Let's look at the $k$ steps made by the `gcd0`$(m, n)$:

$$\begin{aligned} m &= q_1 n + r_1 \\ n &= q_2 r_1 + r_2 \\ r_1 &= q_3 + r_3 \\ &\vdots \\ r_{k-2} &= q_k r_{k-1} + r_k \\ r_{k-1} &= q_{k+1} r_k \end{aligned}$$

Using the previous Lemma we know that,

$$\gcd(F_m, F_n) = \gcd(F_{r_1}, F_n) = \gcd(F_{r_1}, F_{r_2}) = \ldots = \gcd(F_{r_{k-1}}, F_{r_k})$$

Because we know that $r_k \mid r_{k-1}$, we have $F_{r_k} \mid F_{r_{k+1}}$. So,

$$\gcd(F_m, F_n) = \gcd(F_{r_{k-1}}, F_{r_k}) = F_{r_k} = F_{\gcd(n,m)}$$

□

**Exercise 12.** In Example 9 you proved that

$$F_n F_{n-1} = F_n^2 - F_{n-1}^2 + (-1)^n.$$

Show why this implies that consecutive Fibonacci numbers are relatively prime.

**Exercise 13.** Evaluate $\gcd(F_9, F_{12})$, $\gcd(F_{15}, F_{20})$ and $\gcd(F_{24}, F_{36})$.

**Example 12.** Show that the sequence of $\text{rem}(F_n, 2)$ repeats periodically.

*Proof.*Let's look at the first Fibonacci values and their remainder divided by 2.

| $F_1$ | $F_2$ | $F_3$ | $F_4$ | $F_5$ | $F_6$ | $F_7$ | $F_8$ | $F_9$ | $F_{10}$ |
|---|---|---|---|---|---|---|---|---|---|
| 1 | 1 | 2 | 3 | 5 | 8 | 13 | 21 | 34 | 55 |
| 1 | 1 | 0 | 1 | 1 | 0 | 1 | 1 | 0 | 1 |

It looks like they follow the pattern $1, 1, 0$. *Why?* Let's see what happens when we are at $1, 1$. Assume $F_n = 2k_1 + 1$ and $F_{n+1} = 2k_2 + 1$, then $F_{n+2} = F_{n+1} + F_n = 2k_1 + 1 + 2k_2 + 1 = 2(k_1 + k_2 + 1) + 0$. So, we get a 0. What happens if we are at $1, 0$? We get $\text{rem}(F_{n+2}, 2) = 1$. So the next remainder only depends on the previous two, so it will create the same pattern again and again. □

Actually, the same holds for any fixed modulo. Take for example 4. The cycle here is a bit longer but it eventually repeats.

| $F_1$ | $F_2$ | $F_3$ | $F_4$ | $F_5$ | $F_6$ | $F_7$ | $F_8$ | $F_9$ | $F_{10}$ | $F_{11}$ | $F_{12}$ | $F_{13}$ | $F_{14}$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 1 | 2 | 3 | 5 | 8 | 13 | 21 | 34 | 55 | 89 | 144 | 233 | 377 |
| 1 | 1 | 2 | 3 | 1 | 0 | 1 | 1 | 2 | 3 | 1 | 0 | 1 | 1 |

**Exercise 14.** Show that the sequence of $\text{rem}(F_n, k)$ repeats periodically for any constant $k$. Show that this period has length at most $k^2 - 1$. *Hint:* Consider all possible pairs of remainders for $F_n$ and $F_{n+1}$, and use pigeonhole principle.

**Note:** The lengths of these periods are known as *Pisano periods*. You can find more on the wikipedia page. Finding a closed-form solution for Pisano periods is an open problem (see here).

**Example 13.** Establish that $F_{n+3} \equiv F_n \pmod 2$ and deduce that $F_3, F_6, F_9, \ldots$.

*Proof.* This exercise is asking us to prove what we showed above in a different way. We will use expansion for the first part,

$$F_{n+3} = F_{n+2} + F_{n+1} = (F_{n+1} + F_n) + F_{n+1} = 2F_{n+1} + F_n \equiv F_n \pmod n.$$

□

**Exercise 15.** Show that $F_{n+5} \equiv 3F_n \pmod 5$ and deduce that $F_5, F_{10}, F_{15}, \ldots$ are divisible by 5. *Hint:* You will need to use expansion. Notice the order of expansion in the previous example to avoid having too many.

**Exercise 16.** Use the fact that $m \mid n$ implies $F_m \mid F_n$ to verify the assertions below:

($a$) $2 \mid F_n$ iff $3 \mid n$.

($b$) $3 \mid F_n$ iff $4 \mid n$.

($c$) $5 \mid F_n$ iff $5 \mid n$.

($d$) $8 \mid F_n$ iff $6 \mid n$.

**Example 14.** Show that if $2 \mid F_n$, then $4 \mid (F_{n+1}^2 - F_{n-1}^2)$.

*Proof.* Since $2 \mid n$ and $2 \mid n+2$, $F_2 = 2 \mid F_{n+2}$.

$$F_{n+1}^2 - F_{n-1}^2 = (F_{n+1} - F_{n-1})(F_{n+1} + F_{n-1}) = F_n \cdot F_{n+2}$$

Hence, this is a product of two even numbers so divisible by 4. □

**Exercise 17.** Use induction to show that $F_{2n} \equiv n(-1)^{n+1} \pmod 5$ for $n \geq 1$.

# Past papers

### COMPUTER SCIENCE TRIPOS Part IA – 2004 – Paper 1

**7 Discrete Mathematics (MPF)**

Recall the Fibonacci numbers defined by:

$$\begin{aligned} f_0 &= 0 \\ f_1 &= 1 \\ f_n &= f_{n-1} + f_{n-2} \text{ for } n > 1 \end{aligned}$$

Using induction on $n$, or otherwise, show that $f_{m+n} = f_{m-1}f_n + f_m f_{n+1}$ for $m > 0$. [4 marks]

Deduce that $\forall m, n > 0 \,.\, m|n \Rightarrow f_m|f_n$. [4 marks]

Deduce further that $\forall n > 4 \,.\, f_n$ prime $\Rightarrow n$ prime. [2 marks]

Given $n \in \mathbb{N}$, let $g_i = f_i \bmod n$, and consider the pairs $(g_1, g_2)$, $(g_2, g_3), \ldots, (g_i, g_{i+1}), \ldots$ . Show that there must be a repetition in the first $n^2 + 1$ pairs. Let $r < s$ be the least values with $(g_r, g_{r+1}) = (g_s, g_{s+1})$. Show that $g_{r-1} = g_{s-1}$, and deduce that $r = 1$. Calculate $g_1$ and $g_2$, and deduce that $g_{s-1} = 0$. Hence show that one of the first $n^2$ Fibonacci numbers is divisible by $n$.

[10 marks]

COMPUTER SCIENCE TRIPOS Part IA – 2018 – Paper 2

**9 Discrete Mathematics (MPF)**

($a$) Define $F_0 = 0$, $F_1 =!$ and for $n \in \mathbb{N}$, $F_{n+2} = F_{n+1} + F_n$.

For positive integers $a$ and $b$, prove that

$$\forall n \in \mathbb{N}.\, \gcd(aF_{n+3} + bF_{n+2}, aF_{n+1} + bF_n) = \gcd(a, b)$$

# (optional) Connection to matrices

**Theorem 3.** The Fibonacci numbers satisfy the following recurrence:

$$\begin{bmatrix} F_{n+2} \\ F_{n+1} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} F_{n+1} \\ F_n \end{bmatrix}$$

**Theorem 4.** Show that

$$\begin{bmatrix} F_{n+1} \\ F_n \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^n \begin{bmatrix} F_1 \\ F_0 \end{bmatrix}$$

**Theorem 5.** Show that

$$\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^n = \begin{bmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{bmatrix}$$

**Example 15.** Prove using the previous relation that for all $n$ and $m$,

$$F_{n+m+1} = F_{n+1}F_{m+1} + F_nF_m.$$

*Proof.*We will use the fact that $M^{n+m} = M^n \cdot M^m$ by associativity of multiplication. The top left entry in the LHS is $F_{n+m+1}$, so we just need to find what is this equal to on the RHS,

$$M^n \cdot M^m = \begin{bmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{bmatrix} \cdot \begin{bmatrix} F_{m+1} & F_m \\ F_m & F_{m-1} \end{bmatrix} = \begin{bmatrix} F_{n+1}F_{m+1} + F_nF_m & \dots \\ \dots & \dots \end{bmatrix}$$

Since the two matrices are equal, the top left entries are equal so $F_{n+m+1} = F_{n+1}F_{m+1} + F_nF_m$. $\square$

**Theorem 6.** Let $D = \begin{bmatrix} d_{11} & 0 \\ 0 & d_{22} \end{bmatrix}$ be a diagonal $2 \times 2$ matrix with entries along the diagonal, then for any positive integer $n$, $D^n = \begin{bmatrix} d_{11}^n & 0 \\ 0 & d_{22}^n \end{bmatrix}$.

**Theorem 7.** Let $M = UDU^{-1}$ where $M$, $U$ and $D$ are $2 \times 2$ matrices and $D$ is diagonal. Show that $M^n = UD^nU^{n-1}$

*Hint:* For $n = 3$, $M^3 = (UDU^{-1})(UDU^{-1})(UDU^{-1}) = UD(U^{-1}U)D(U^{-1}U)DU^{-1} = UD^3U^{-1}$

**Theorem 8.** Show using eigenvector decomposition or otherwise that

$$\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} = \underbrace{\begin{bmatrix} \frac{1}{2}(1-\sqrt{5}) & \frac{1}{2}(1+\sqrt{5}) \\ 1 & 1 \end{bmatrix}}_{U} \underbrace{\begin{bmatrix} \frac{1}{2}(1-\sqrt{5}) & 0 \\ 0 & \frac{1}{2}(1+\sqrt{5}) \end{bmatrix}}_{D} \underbrace{\begin{bmatrix} -\frac{1}{\sqrt{5}} & \frac{1}{10}(5+\sqrt{5}) \\ \frac{1}{\sqrt{5}} & \frac{1}{10}(5-\sqrt{5}) \end{bmatrix}}_{V}$$

and $V = U^{-1}$.

**Exercise 18.** Using Theorem 7, 6 and 8, prove the closed-formed formula for the Fibonacci numbers.

# (optional) Further questions/references

You will find more related material in the following:

- Look at other methods for solving linear recurrences such as using generating functions.
- Look at Chapter 14 in Elementary Number Theory by Burton for more problems.
- Look at the Fibonacci Quarterly archive (containing several problem lists).

For more algorithmic material on the Fibonacci numbers:

- Look at the Fibonacci heaps in Part IA algorithms.
- Loot at the Fibonacci trees used for balls and bins analysis in randomised algorithms (possibly Part II Probability and Computing).

## (optional) Zeckendorf's theorem

**Exercise 19.** $(a)$ Prove that every non-negative integer can be written as the sum of distinct, nonconsecutive Fibonacci numbers. That is, if the Fibonacci number $F_i$ appears in the sum, it appears exactly once, and its neighbors $F_{i-1}$ and $F_{i+1}$ do not appear at all. For example:

$$
\begin{aligned}
17 &= F_7 + F_4 + F_2 \\
42 &= F_9 + F_6 \\
54 &= F_9 + F_7 + F_5 + F_3
\end{aligned}
$$

(*b*) Prove that every positive integer can be written as the sum of distinct Fibonacci numbers with no consecutive gaps. That is, for any index $i \geq 1$, if the consecutive Fibonacci numbers $F_i$ or $F_{i+1}$ do not appear in the sum, then no larger Fibonacci number $F_j$ with $j > i$ appears in the sum. In particular, the sum must include either $F_1$ or $F_2$. For example:

$$
\begin{aligned}
16 &= F_6 + F_5 + F_3 + F_2 \\
42 &= F_8 + F_7 + F_5 + F_3 + F_1 \\
54 &= F_8 + F_7 + F_6 + F_5 + F_4 + F_3 + F_2 + F_1
\end{aligned}
$$

**Source**: Exercise 7 from this handout.

**Further reading:** This theorem finds application in a 2-player game, the *Fibonacci Nim* game. (You may want to read Exercise 7 on page 7 in "Game Theory" by T. Ferguson and see this applet)