

Intro to Probability

Solution Notes for Example Sheet 1

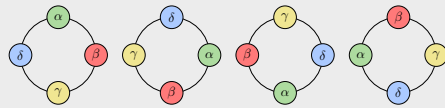
1 Counting/Combinatorics

Exercise 1 How many ways are there to assign N workers to N jobs such that each worker is assigned one job? (If it helps, you may assume $N = 10$)

(Answer) The first job can be assigned to any of the N workers, the second job to any of the $N - 1$ remaining workers, and so on. Hence, in total we multiply the number of ways (using the product rule) to obtain:

$$N \cdot (N - 1) \cdot \dots \cdot 2 \cdot 1 = N!$$

Exercise 2 How many ways are there to place N people on a circular table with N seats? For instance, the following four configurations are considered the same for $N = 4$.



(Answer) There are $N!$ ways to place the N people on the N seats. Each seating configuration is equivalent to its N cyclic rotations. Hence, out of the $N!$ ways, there are $N!/N = (N - 1)!$ unique ways.

Exercise 3 How many ways are there to assign N workers to M jobs? (If it helps, you may assume $N = 10$ and $M = 14$)

(Answer) Each of the M jobs can be assigned to one of the N workers. So using the product rule, there are N^M possible assignments.

Exercise 4 For years, telephone area codes in the United States and Canada consisted of a sequence of three digits. The first digit was an integer between 2 and 9, the second digit was either 0 or 1, and the third digit was any integer from 1 to 9. How many area codes were possible? How many area codes starting with a 4 were possible?

[Source: Ross P1.5]

(Answer)

$$8 \cdot 2 \cdot 9 = 144.$$

Exercise 5 In how many ways can 8 people be seated in a row if

- there are no restrictions on the seating arrangement?
- persons A and B must sit next to each other?
- there are 4 men and 4 women and no 2 men or 2 women can sit next to each other?
- there are 5 men and they must sit next to one another?
- there are 4 married couples and each couple must sit together?

[Source: Ross P1.10]

(Answer)

- There are $8!$ ways.
- By considering AB as a single block there are $7!$ ways of arranging them and then twice that for the ordering of AB and BA , so $2 \cdot 7!$ in total.

- (c) We can either have $MWMMWMMW$ or $WMWMMWMM$ and then we can fill these with $(4!)^2$ ways. So, $2 \cdot (4!)^2$ ways in total.
- (d) By considering the 5 men as a block, then there are $4!$ ways of ordering them and each one has $5!$ ways of being ordered. So, in total there are $5! \cdot 4!$.
- (e) There are $4!$ ways of ordering the couples and 2 ways to order each couple. So $2 \cdot 4!$ ways in total.

Exercise 6 A dance class consists of 22 students, of which 10 are women and 12 are men. If 5 men and 5 women are to be chosen and then paired off, how many results are possible?

[Source: Ross P1.15]

(Answer) There are $\binom{10}{5}$ ways of choosing the 5 women and $12 \cdot 11 \cdot 10 \cdot 9 \cdot 8$ ways of selecting their pairs. So their product is the total valid configurations.

Exercise 7

- (a) Prove that

$$\binom{n+m}{r} = \binom{n}{0} \binom{m}{r} + \binom{n}{1} \binom{m}{r-1} + \dots + \binom{n}{r} \binom{m}{0}.$$

[Hint: Consider a group of n men and m women. How many groups of size r are possible?]

- (b) Use part (a) to show that

$$\binom{2n}{n} = \sum_{k=0}^n \binom{n}{k}^2.$$

[Source: Ross T1.8-9]

(Answer)

- (a) In order to form a group of size r , we can select 0 men and r women, or 1 men and $r-1$ women, ... , or r men and 0 women. By summing over all the cases, we have that

$$\binom{n+m}{r} = \binom{n}{0} \binom{m}{r} + \binom{n}{1} \binom{m}{r-1} + \dots + \binom{n}{r} \binom{m}{0}.$$

- (b) By choosing $m = r = n$, we get

$$\binom{2n}{n} = \binom{n+n}{n} = \binom{n}{0} \binom{n}{n} + \binom{n}{1} \binom{n}{n-1} + \dots + \binom{n}{n} \binom{n}{0}.$$

Using that for any k , $\binom{n}{k} = \binom{n}{n-k}$, we have that

$$\binom{2n}{n} = \binom{n+n}{n} = \binom{n}{0} \binom{n}{0} + \binom{n}{1} \binom{n}{1} + \dots + \binom{n}{n} \binom{n}{n} = \sum_{k=0}^n \binom{n}{k}^2.$$

Exercise 8 From a group of n people, suppose that we want to choose a committee of k , $k \leq n$, one of whom is to be designated as chairperson.

- (a) By focusing first on the choice of the committee and then on the choice of the chair, argue that there are $\binom{n}{k} \cdot k$ possible choices.
- (b) By focusing first on the choice of the nonchair committee members and then on the choice of the chair, argue that there are $\binom{n}{k-1} \cdot (n-k+1)$ possible choices.
- (c) By focusing first on the choice of the chair and then on the choice of the other committee members, argue that there are $n \cdot \binom{n-1}{k-1}$ possible choices.
- (d) Conclude from parts (1), (2), and (3) that

$$\binom{n}{k} \cdot k = \binom{n}{k-1} \cdot (n-k+1) = n \cdot \binom{n-1}{k-1}.$$

(e) Use the factorial definition of $\binom{m}{r}$ to verify the identity in part (4).

[Source: Ross T1.10]

Exercise 9 Enumerate all distinct sequences of length 4 consisting of symbols A, A, B, C .

(Answer) There are the following sequences:

AABC
AACB
ABAC
ABCA
ACAB
ACBA
BAAC
BACA
BCAA
CAAB
CABA
CBAA

We can verify that their number is equal to:

$$\frac{4!}{2!1!1!} = 12.$$

Exercise 10 You are playing bridge. When you pick up your hand, you notice that the suits are already grouped; that is, the clubs are all adjacent to each other, the hearts likewise, and so on. Given that your hand contains four spades, four hearts, three diamonds, and two clubs, what is the probability $\Pr[(\cdot)G]$ of this event G ?

[Source: Elementary probability 3.2.5]

(Answer) The number of ways to arrange all 13 cards is $13!$ and the cards within each suit is $4!, 4!, 3!$ and $2!$ respectively. There are also $4!$ ways of arranging the suits. Hence, the probability of the event G is

$$\Pr[G] = \frac{(4!)^3 \cdot 3! \cdot 2!}{13!}.$$

2 Axioms/Probability spaces

Exercise 11 Let E and F be two events for which one knows that the probability that at least one of them occurs is $3/4$. What is the probability that neither E nor F occurs? [Hint: Use one of DeMorgan's laws: $E^c \cap F^c = (E \cup F)^c$.]

[Source: Dekking 2.2]

(Answer) We are given that

$$\Pr[E \cup F] = \frac{3}{4}.$$

Then, we have that

$$\Pr[\neg(E \cup F)^c] = 1 - \Pr[E \cup F] = 1 - \frac{3}{4} = \frac{1}{4}.$$

Exercise 12 Let C and D be two events for which one knows that $\Pr[C] = 0.3$, $\Pr[D] = 0.4$, and $\Pr[C \cap D] = 0.2$. What is $\Pr[C^c \cap D]$?

[Source: Dekking 2.3]

(Answer)

$$\Pr[C^c \cap D] = \Pr[D] - \Pr[C \cap D] = 0.4 - 0.2 = 0.2.$$

Exercise 13 We consider events A , B , and C , which can occur in some experiment. Is it true that the probability that only A occurs (and not B or C) is equal to $\Pr[A \cup B \cup C] - \Pr[B] - \Pr[C] + \Pr[B \cap C]$?

[Source: Dekking 2.4]

(Answer)

$$\begin{aligned} & \Pr[A \cup B \cup C] - (\Pr[B] + \Pr[C] - \Pr[B \cap C]) \\ & \Pr[A \cup B \cup C] - \Pr[B \cup C] \\ & \Pr[A \cap (B \cup C)^c] \\ & \Pr[A \cap B^c \cap C^c]. \end{aligned}$$

Exercise 14 When $\Pr[A] = 1/3$, $\Pr[B] = 1/2$, and $\Pr[A \cup B] = 3/4$, what is

- (a) $\Pr[A \cap B]$?
- (b) $\Pr[A^c \cup B^c]$?

[Source: Dekking 2.6]

(Answer)

- (a) $\Pr[A \cap B] = \Pr[A] + \Pr[B] - \Pr[A \cup B] = 1/3 + 1/2 - 3/4 = 1/12$.
- (b) Using that $(A^c \cup B^c) = (A \cap B)^c$, we have that

$$\Pr[A^c \cup B^c] = 1 - \Pr[A \cap B] = 11/12.$$

Exercise 15 For any sequence of events E_1, E_2, \dots , define a new sequence F_1, F_2, \dots of disjoint events (that is, events such that $F_i \cap F_j = \emptyset$ whenever $i \neq j$) such that for all $n \geq 1$,

$$\cup_{i=1}^n F_i = \cup_{i=1}^n E_i.$$

[Source: Ross T2.5]

(Answer) One way to do this is to define $F_1 = E_1$ and for any $i \geq 2$,

$$F_i = E_i \setminus \bigcup_{j=1}^{i-1} E_j.$$

Intuitively, F_i has the items that were not seen in events $1, \dots, i-1$. This way $F_i \cap F_j = \emptyset$ for $i \neq j$ and also

$$\cup_{i=1}^n F_i = \cup_{i=1}^n E_i.$$

Exercise 16 Prove that $\Pr[E \cap \bar{F}] = \Pr[E] - \Pr[E \cap F]$. **Exercise 17** [Ross T2.13]

(Answer) By rearranging, we just need to show that

$$\Pr[E] = \Pr[E \cap F] + \Pr[E \cap \bar{F}],$$

which follows from the law of total probability.

Exercise 18 Show that the probability that exactly one of the events E or F occurs equals $\Pr[E] + \Pr[F] - 2\Pr[E \cap F]$.

[Source: Ross T2.12]

(Answer) By Exercise 16, we have that

$$\Pr[E] = \Pr[E \cap F] + \Pr[E \cap \bar{F}], \quad \text{and} \quad \Pr[F] = \Pr[F \cap E] + \Pr[F \cap \bar{E}].$$

By adding these two we have that

$$\Pr[E \cap \bar{F}] + \Pr[F \cap \bar{E}] = \Pr[E] + \Pr[F] - 2\Pr[E \cap F].$$

Exercise 19 [Bonferroni inequalities]

(a) Prove that for any two events E and F : $\Pr[E \cap F] \geq \Pr[E] + \Pr[F] - 1$.

(b) (Optional) Generalise to n events: $P(E_1 \cap \dots \cap E_n) \geq P(E_1) + \dots + P(E_n) - (n - 1)$.

(Answer)

(a)

$$\Pr[E \cap F] = \Pr[E] + \Pr[F] - \Pr[E \cup F] \geq \Pr[E] + \Pr[F] - 1.$$

Exercise 20 Consider an experiment whose sample space consists of a countably infinite number of points. Show that not all points can be equally likely. Can all points have a positive probability of occurring?

3 Conditional probability and Bayes' theorem

Exercise 21 Show that for events E and F ,

$$\frac{\Pr[H | E]}{\Pr[H^c | E]} = \frac{\Pr[E | H] \Pr[H]}{\Pr[E | H^c] \Pr[H^c]}.$$

(Answer) By Bayes' rule we have that

$$\Pr[H | E] = \frac{\Pr[E | H] \cdot \Pr[H]}{\Pr[E]} \quad \text{and} \quad \Pr[H^c | E] = \frac{\Pr[E | H^c] \cdot \Pr[H^c]}{\Pr[E]}.$$

By taking the ratio of the two probabilities we have that

$$\frac{\Pr[H | E]}{\Pr[H^c | E]} = \frac{\Pr[E | H] \Pr[H]}{\Pr[E | H^c] \Pr[H^c]}.$$

Exercise 22 Consider 3 urns. Urn A contains 2 white and 4 red balls, urn B contains 8 white and 4 red balls, and urn C contains 1 white and 3 red balls. If 1 ball is selected from each urn, what is the probability that the ball chosen from urn A was white given that exactly 2 white balls were selected?

[Source: Ross T3.9]

Exercise 23 A fair die is thrown twice. A is the event “sum of the throws equals 4,” B is “at least one of the throws is a 3.”

(a) Calculate $\Pr[A | B]$.

(b) Are A and B independent events?

[Source: Dekking 3.2]

(Answer)

(a)

$$\Pr[A | B] = \frac{\Pr[A, B]}{\Pr[B]} = \frac{2/36}{11/36} = \frac{2}{11}$$

(b) We have that

$$\Pr[A] = \frac{3}{36} \neq \Pr[A | B],$$

so the events A and B are not independent.

Exercise 24 A ball is in any one of n boxes and is in the i th box with probability P_i . If the ball is in box i , a search of that box will uncover it with probability α_i . Show that the conditional probability that the ball is in box j , given that a search of box i did not uncover it, is

(a) $\frac{P_j}{1 - \alpha_i P_i}$ for $i \neq j$.

(b) $\frac{(1 - \alpha_i)P_i}{1 - \alpha_i P_i}$ for $i = j$.

[Source: Dekking 3.4]

(Answer) Let A_j be the event that the ball was in box j and B_i the event that the ball was not found in box i . Then when $i \neq j$,

$$\Pr[A_j | B_i] = \frac{\Pr[B_i | A_j] \cdot \Pr[A_j]}{\Pr[B_i]} = \frac{P_j \cdot 1}{1 - P_i \alpha_i}.$$

Similarly, for $i = j$,

$$\Pr[A_j | B_j] = \frac{P_j \cdot (1 - \alpha_j)}{1 - P_j \alpha_j}.$$

Exercise 25 [Extended multiplication rule]

(a) Show that $\Pr[A, B, C] = \Pr[A|B, C] \Pr[B|C] \Pr[C]$.

(b) Show that $\Pr[A_1, \dots, A_n] = \Pr[A_1|A_2, \dots, A_n] \cdot \dots \Pr[A_{n-1}|A_n] \Pr[A_n]$.

(Answer)

(a) Starting with the RHS and applying the definition of conditional probability:

$$\Pr[A|B, C] \Pr[B|C] \Pr[C] = \Pr[A|B, C] \Pr[B, C] = \Pr[A, B, C].$$

(b) We can follow the same steps as in (a) to obtain this generalisation.

4 Independence

Exercise 26 Suppose that E and F are mutually exclusive events of an experiment. Show that if independent trials of this experiment are performed, then E will occur before F with probability

$$\frac{\Pr[E]}{\Pr[E] + \Pr[F]}.$$

[Source: Ross T3.76]

(Answer) The probability that event E occurs for the first time at step i and in the previous steps neither has occurred is given by,

$$(1 - \Pr[E] - \Pr[F])^{i-1} \cdot \Pr[E].$$

The probability of E occurring before F is given by

$$\begin{aligned} & \sum_{i=1}^{\infty} (1 - \Pr[E] - \Pr[F])^{i-1} \cdot \Pr[E] \\ &= \Pr[E] \cdot \sum_{i=0}^{\infty} (1 - \Pr[E] - \Pr[F])^i \\ &= \Pr[E] \cdot \frac{1}{1 - (1 - \Pr[E] - \Pr[F])} \\ &= \frac{\Pr[E]}{\Pr[E] + \Pr[F]}. \end{aligned}$$

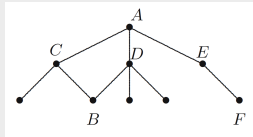
Exercise 27 [Decomposition] (Optional) Let X, A, B be random variables. Show that if X is independent of (A, B) then it is independent of A and independent of B .

Exercise 28 [Weak union] (Optional) Let X, A, B be random variables. Show that if X is independent of (A, B) , then X is independent of A given B .

Exercise 29 [Concatenation] (Optional) Let X, A, B be random variables. Show that if X is independent of A given B , and X is independent of B , then X is independent of (A, B) .

5 Partition theorem

Exercise 30 Your lecturer wants to walk from A to B (see the map). To do so, he first randomly selects one of the paths to C, D , or E . Next he selects randomly one of the possible paths at that moment (so if he first selected the path to E , he can either select the path to A or the path to F), etc. What is the probability that he will reach B after two selections?



[Source: Dekking 3.1]

Exercise 31 We draw two cards from a regular deck of 52. Let S_1 be the event “the first one is a spade”, and S_2 “the second one is a spade”.

- Compute $\Pr[S_1]$, $\Pr[S_2 | S_1]$, and $\Pr[S_2 | S_1^c]$.
- Compute $\Pr[S_2]$ by conditioning on whether the first card is a spade.

[Source: Dekking 3.3]

(Answer)

- $\Pr[S_1] = 1/4$, $\Pr[S_2 | S_1] = 12/51$ and $\Pr[S_2 | S_1^c] = 13/51$.
- Using the partition theorem,

$$\Pr[S_2] = \Pr[S_2 | S_1] \cdot \Pr[S_1] + \Pr[S_2 | S_1^c] \cdot \Pr[S_1^c] = \frac{12}{51} \cdot \frac{1}{4} + \frac{13}{51} \cdot \frac{3}{4} = \frac{1}{4},$$

as expected.

6 Principle of inclusion exclusion

Exercise 32 [Derrangements] A *derrangement* is a permutation p , such that $p(i) \neq i$ for all i . In this exercise, you will derive the formula for the number of derrangements of length n .

- (a) By defining E_i the event that the $i = p(i)$ maps to itself and using the principle of inclusion-exclusion, show that the number of derrangements of length n is

$$D_n = n! \sum_{k=1}^n (-1)^k \frac{1}{k!}.$$

- (b) Show that as $n \rightarrow \infty$, $\frac{1}{n!}D_n$ approaches $1/e$.

Exercise 33 (Optional) Compute the number of multiples of 2 or 3 from 1 to 1000.

Exercise 34 (Optional) Count the number of sequences of length n , consisting only of numbers 0, 1 and 2, such that each number occurs at least once.

Exercise 35 There are n distinct types of coupons, and each coupon obtained is, independently of prior types collected, of type i with probability p_i , $\sum_{i=1}^n p_i = 1$.

- (a) If n coupons are collected, what is the probability that one of each type is obtained?
- (b) Now suppose that $p_1 = p_2 = \dots = p_n = 1/n$. Let E_i be the event that there are no type i coupons among the n collected. Apply the inclusion-exclusion identity for the probability of the union of events to $\Pr[\cup_i E_i]$ to prove the identity

$$n! = \sum_{k=0}^n (-1)^k \binom{n}{k} (n-k)^n.$$

7 Discrete random variables

Exercise 36

- (a) What is a random variable?
- (b) What is a discrete random variable?
- (c) How is the expectation of a random variable defined? Is it always finite?

Exercise 37 [CDF]

- (a) What are the properties of the PMF function.
- (b) What are the properties of a CDF graph.
- (c) For the example on Lecture 2 Slide 4, draw the graphs of the PMF and the CDF.

Exercise 38 Suppose that a die is rolled twice. What are the possible values that the following random variables can take on:

- (a) the maximum value to appear in the two rolls;
- (b) the minimum value to appear in the two rolls;
- (c) the sum of the two rolls;
- (d) the value of the first roll minus the value of the second roll?
- Assuming a fair die, what is the probability for each event?

[Source: Ross P4.7-4.8]

Exercise 39 [St. Petersburg paradox] A person tosses a fair coin until a tail appears for the first time. If the tail appears on the n th flip, the person wins 2^n dollars. Let X denote the player's winnings. Show that $E[X] = \infty$.

- (a) Would you be willing to pay \$1 million to play this game once?
- (b) Would you be willing to pay \$1 million for each game if you could play for as long as you liked and only had to settle up when you stopped playing?

[Source: Ross P4.30]