Example sheet 1

Learning with probability models
Data Science—DJW—2020/2021

Question 1. Sketch the cumulative distribution function, and calculate the density function, for this continuous random variable:

```
def rx():
    u = random.random()
    return u * (1-u)
```

Question 2. Given a dataset $(x_1, \dots, x_n)$, we wish to fit a Poisson distribution. This is a discrete random variable with a single parameter $\lambda > 0$, called the rate, and

$$\Pr(x\,;\,\lambda) = \frac{\lambda^x e^{-\lambda}}{x!} \quad \text{for } x \in \{0, 1, 2, \dots\}.$$

Show that the maximum likelihood estimator for λ is $\hat{\lambda} = n^{-1} \sum_{i=1}^{n} x_i$.

Question 3. Given a dataset [3,2,8,1,5,0,8], we wish to fit a Poisson distribution. Give code to achieve this fit, using `scipy.optimize.fmin`.

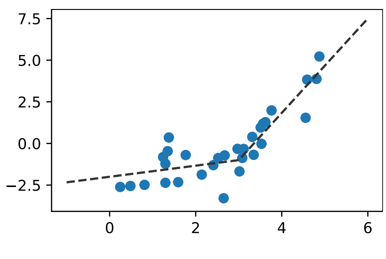
Question 4. Given a dataset $(x_1, \dots, x_n)$, we wish to fit the Uniform$[0, \theta]$ distribution, where θ is unknown. Show that the maximum likelihood estimator is $\hat{\theta} = \max_i x_i$.

Question 5 (A/B testing). Your company has two systems which it wishes to compare, A and B. It has asked you to compare the two, on the basis of performance measurements $(x_1, \dots, x_m)$ from system A and $(y_1, \dots, y_n)$ from system B. Any fool using Excel can just compare the averages, $\bar{x} = m^{-1} \sum_{i=1}^{m} x_i$ and $\bar{y} = n^{-1} \sum_{i=1}^{n} y_i$, but you are cleverer than that and you will harness the power of Machine Learning.

Suppose the x_i are drawn from $X \sim \text{Normal}(\mu, \sigma^2)$, and the y_i are drawn from $Y \sim \text{Normal}(\mu + \delta, \sigma^2)$, and all the samples are independent, and μ, δ, and σ are unknown. Find maximum likelihood estimators for the three unknown parameters.

Question 6. Let x_i be the population of city i, and let y_i be the number of crimes reported. Consider the model $Y_i \sim \text{Poisson}(\lambda x_i)$, where $\lambda > 0$ is an unknown parameter. Find the maximum likelihood estimator $\hat{\lambda}$.

Question 7. We wish to fit a piecewise linear line to a dataset, as shown below. The inflection point is given, and we wish to estimate the slopes and intercepts. Explain how to achieve this using a linear modelling approach.



*Note. As a sanity check, you should implement your model formula as a function and plot it. Here's a function that **fails** the check.*

```
def pred(x, m₁,c₁,m₂,c₂, inflection_x=3):
    e = numpy.where(x <= inflection_x, 1, 0)
    return e*(m₁*x + c₁) + (1-e)*(m₂*x+c₂)
x = numpy.linspace(0,5,1000)
plt.plot(x, pred(x, m₁=0.5,c₁=0,m₂=1,c₂=2))
```

Question 8. For the climate data from section 2.2.5 of lecture notes, we proposed the model

$$\texttt{temp} \approx \alpha + \beta_1 \sin(2\pi \texttt{t}) + \beta_2 \cos(2\pi \texttt{t}) + \gamma \texttt{t}$$

in which the $+\gamma\texttt{t}$ term asserts that temperatures are increasing at a constant rate. We might suspect though that temperatures are increasing non-linearly. To test this, we can create a non-numerical feature out of `t` by

```
u = 'decade_' + str(math.floor(t/10)) + '0s'
```

(which gives us values like `'decade_1980s'`, `'decade_1990s'`, etc.) and fit the model

$$\texttt{temp} \approx \alpha + \beta_1 \sin(2\pi \texttt{t}) + \beta_2 \cos(2\pi \texttt{t}) + \gamma_{\texttt{u}}.$$

Write this as a linear model, and give code to fit it. *[Note. You should explain what your feature vectors are, then give a one-line command to estimate the parameters.]*

Question 9. I have two feature vectors

$$\texttt{gender} = [f, f, f, f, m, m, m], \qquad \texttt{eth} = [a, a, b, w, a, b, b]$$

and I one-hot encode them as

$$g_1 = [1, 1, 1, 1, 0, 0, 0] \qquad e_1 = [1, 1, 0, 0, 1, 0, 0]$$
$$g_2 = [0, 0, 0, 0, 1, 1, 1] \qquad e_2 = [0, 0, 1, 0, 0, 1, 1]$$
$$e_3 = [0, 0, 0, 1, 0, 0, 0]$$

Are these five vectors $\{g_1, g_2, e_1, e_2, e_3\}$ linearly independent? If not, find a linearly independent set of vectors that spans the same feature space.

Question 10. For the police stop-and-search dataset in section 2.6, we wish to investigate intersectionality in police bias. We propose the linear model

$$1[\texttt{outcome="find"}] \approx \alpha_{\texttt{gender}} + \beta_{\texttt{eth}}.$$

Write this as a linear model using one-hot coding. Are the parameters identifiable? If not, rewrite the model so they are, and interpret the parameters of your model.

[Optional.] Fit the model and report your findings. Code to read the data and prepare `eth` and `gender` features can be found at `https://github.com/damonjw/datasci/blob/master/stop-and-search.ipynb`.

Hints and comments

Question 1. See the example from section 1.5. Sketch a graph of $u(1-u)$ as a function of u. For what ranges of u is $u(1-u) \leq y$? What is the probability that the random variable $U \sim U[0,1]$ lies in these ranges?

Question 2. This is a question about learning generative models. See section 1.6.

Question 3. See section 1.2. What parameter transform is needed here? Also, if you use numpy, watch out for which variables in your code are vectors and which are scalars.

Question 4. This is a question about generative models, section 1.6. You will also need to use the indicator function trick, from section 1.1 exercise 1.4.

Question 5. You can treat this as a pure example of maximum likelihood estimation, as stated in section 1.1 and repeated in section 1.5. You are maximizing $\Pr(\text{data}; \text{params})$, where 'data' should include absolutely all data that can shed light on the params. The data here is $(x_1, \ldots, x_m, y_1, \ldots, y_n)$, and the params are (μ, δ, σ). Don't try to estimate μ from the x_i alone.

You can also treat this along the lines of section 2.4, as a linear model with a probabilistic interpretation. See the discussion in section 2.2 about designing features to compare groups.

Question 6. Supervised learning. See section 1.7 and the example of binomial regression.

Question 7. See section 2.2 on feature design, and the example of a step function.

Question 8. See section 2.2 on feature design, and the subsection on step function responses.

Question 9. See section 2.5 and the exercise about finding a linearly independent subset.

Supplementary question sheet 1

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These questions are not intended for supervision (unless your supervisor directs you otherwise). Some of them are longer form exam-style questions, which you can use for revision. Some others, labelled $*$, ask you to think outside the box.

Question 11 (Cardinality estimation).

(a) Let T be the maximum of m independent Uniform$[0,1]$ random variables. Show that $\mathbb{P}(T \leq t) = t^m$. Find the density function $\Pr_T(t)$. *Hint. For two independent random variables U and V,*

$$\mathbb{P}\big(\max(U,V) \leq x\big) = \mathbb{P}(U \leq x \text{ and } V \leq x) = \mathbb{P}(U \leq x)\,\mathbb{P}(V \leq x).$$

(b) A common task in data processing is counting the number of unique items in a collection. When the collection is too large to hold in memory, we may wish to use fast approximation methods, such as the following: Given a collection of items $a_1, a_2, \ldots$, compute the hash of each item $x_1 = h(a_1), x_2 = h(a_2), \ldots$, then compute $t = \max_i x_i$.

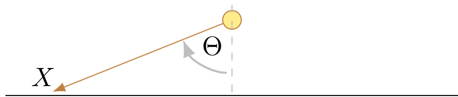
If the hash function is well designed, then each x_i can be treated as if it were sampled from Uniform$[0,1]$, and unequal items will yield independent samples..

The more unique items there are, the larger we expect t to be. Given an observed value t, find the maximum likelihood estimator for the number of unique items. *[Hint. This is about finding the mle from a single observation, as in lecture notes section 1.1.]*

`http://blog.notdot.net/2012/09/Dam-Cool-Algorithms-Cardinality-Estimation`

Question 12. A point lightsource at coordinates $(0,1)$ sends out a ray of light at an angle Θ chosen uniformly in $[-\pi/2, \pi/2]$. Let X be the point where the ray intersects the horizontal line through the origin. What is the density of X? *[Hint. See exercise 3.3, from lecture 2.]*

Note: This random variable is known as the Cauchy distribution. It is unusual in that it has no mean.



Question 13 (Numerical optimization). Fit the model

$$\texttt{Petal.Length} \approx \alpha - \beta(\texttt{Sepal.Length})^\gamma, \qquad \gamma > 0$$

by minimizing the mean square error. *[Hint. This isn't a linear model, so just use `scipy.optimize.fmin`.]*

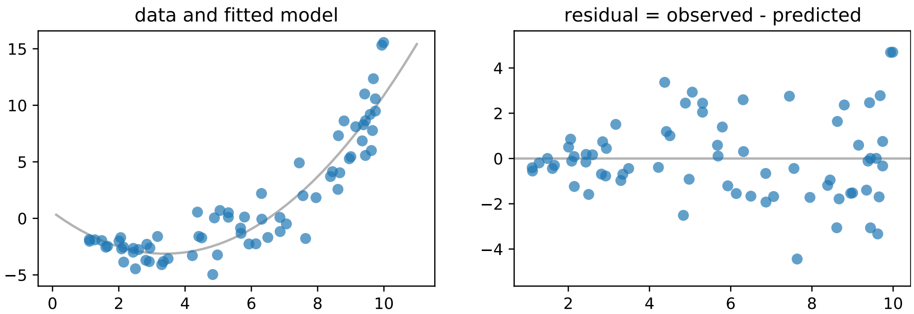
Question 14. As an alternative to the model from question 8, we might suspect that temperatures are increasing linearly up to 1980, and that they are increasing linearly at a different rate from 1980 onwards. Devise a linear model to express this, using your answer to question 7, and fit it. Plot your fit. *[Hint. Sample code for plotting a fit is shown in section 2.1.]*

Question 15 (Heteroscedasticity). We are given a dataset[1] with predictor x and label y and we fit the linear model

$$y_i \approx \alpha + \beta x_i + \gamma x_i^2.$$

After fitting the model using the least squares estimation, we plot the residuals $\varepsilon_i = y_i - (\hat\alpha + \hat\beta x_i + \hat\gamma x_i^2)$.

[1] `https://www.cl.cam.ac.uk/teaching/2021/DataSci/data/heteroscedasticity.csv`



(a) Describe what you would expect to see in the residual plot, if the assumptions behind linear regression are correct.

(b) This residual plot suggests that perhaps $\varepsilon_i \sim \text{Normal}\big(0, (\sigma x_i)^2\big)$ where σ is an unknown parameter. Assuming this is the case, give pseudocode to find the maximum likelihood estimators for α, β, and γ.

[Hint. This question is asking you to reason about a custom probability model, in the style of section 2.4. A model with unequal variances is called 'heteroscedastic'.]

Question 16. Let $(F_1, F_2, F_3, \dots) = (1, 1, 2, 3, \dots)$ be the Fibonacci numbers, $F_n = F_{n-1} + F_{n-2}$. Define the vectors f, f_1, f_2, and f_3 by

$$\begin{aligned} f &= [F_4, F_5, F_6, \dots, F_{m+3}] \\ f_1 &= [F_3, F_4, F_5, \dots, F_{m+2}] \\ f_2 &= [F_2, F_3, F_4, \dots, F_{m+1}] \\ f_3 &= [F_1, F_2, F_3, \dots, F_m] \end{aligned}$$

for some large value of m. If you were to fit the linear model

$$f \approx \alpha + \beta_1 f_1 + \beta_2 f_2$$

what parameters would you expect? What about the linear model

$$f \approx \alpha + \beta_1 f_1 + \beta_2 f_2 + \beta_3 f_3?$$

[Hint. Are the feature vectors linearly independent?]

Question 17*. For the police stop-and-search data from section 2.6, consider the model

$$1[\texttt{outcome} = \text{find}] = \alpha + \sum_{k \neq \text{White}} \beta_k \big(1[\texttt{eth} = k] - 1[\texttt{eth} = \text{White}]\big).$$

Interpret the parameters. *[Hint. What is the predicted value for each ethnicity? What is the average prediction across all ethnicities?]*

Question 18*. Sketch the cumulative distribution functions for these two random variables. Are they discrete or continuous?

```
def rx():
    u = random.random()
    return 1/u
def ry():
    u2 = random.random()
    return rx() + math.floor(u2)
```

[Hint. For intuition, use simulation. Generate say 10,000 samples, and plot a histogram, then a plot of "how many are $\leq x$" as a function of x.]

Question 19*. Is it possible for a continuous random variable to have a probability density function that approaches ∞ at some point in the support? Is it possible to have this and also have finite mean and variance?