Binomial Coefficients for Part IA Discrete Mathematics

Note This handout contains several exercises and past papers relevant to the *Binomial coefficients*.

You can find many more details in Chapter 1.2 of "Elementary number theory" by D. M. Burton, Chapter 4 of "A Walk Through Combinatorics" by M. Bona and in the book "The art of proving binomial identities" by M. Spivey.

Definition

There are a few ways to define the Binomial coefficients: (i) as counting the number of possible sets of size k out of a set of size n, (ii) the recursive definition or (iii) using the factorial formula. In this course, it seems that the definition given is (iii).

Definition 1. The (n, k)-binomial coefficient is defined for natural n, k with $k \leq n$ as

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

Note 1: A trivial observation is that $\binom{n}{0} = 1$.
Note 2: From this definition it is not clear that this is a natural number. The following recurrence relation however confirms this (since the base case is an integer).

Property 1. For all naturals n and k with $k + 1 \leq n$, we have

$$\binom{n+1}{k+1} = \binom{n}{k+1} + \binom{n}{k}$$

*Proof.*Let's expand the RHS,

$$\begin{aligned}
\binom{n}{k+1} + \binom{n}{k} &= \frac{n!}{(k+1)!(n-(k+1))!} + \frac{n!}{k!(n-k)!} \\
&= \frac{n!}{(k+1)!(n-(k+1))!} + \frac{n!}{k!(n-k)!} \\
&= \frac{n!}{k!(n-(k+1))!} \cdot \left(\frac{1}{k+1} + \frac{1}{n-k}\right) \\
&= \frac{n!}{k!(n-(k+1))!} \cdot \frac{n-k+k+1}{(k+1)(n-k)} \\
&= \frac{n!}{k!(n-(k+1))!} \cdot \frac{n+1}{(k+1)(n-k)} \\
&= \frac{(n+1)!}{(k+1)!(n-k)!} \\
&= \frac{(n+1)!}{(k+1)!(n+1-(k+1))!} \\
&= \binom{n+1}{k+1}
\end{aligned}$$

□

This recurrence relation motivates the computation of binomial coefficients using Pascal's triangle, where each entry is the sum of two entries above it.

$$
\begin{array}{lccccccccccccc}
n=0 & & & & & & & 1 & & & & & & \\
n=1 & & & & & & 1 & & 1 & & & & & \\
n=2 & & & & & 1 & & 2 & & 1 & & & & \\
n=3 & & & & 1 & & 3 & & 3 & & 1 & & & \\
n=4 & & & 1 & & 4 & & 6 & & 4 & & 1 & & \\
n=5 & & 1 & & 5 & & 10 & & 10 & & 5 & & 1 & \\
n=6 & 1 & & 6 & & 15 & & 20 & & 15 & & 6 & & 1
\end{array}
$$

Implementation Challenge: Write code to compute Pascal's triangle.

Implementation Challenge: Write code to compute the (n, k)-binomial coefficient using Pascal's triangle. How much time does your algorithm require? Can you do it with $O(n)$ memory?

The above formula also gives the combinatorial viewpoint of Binomial coefficients.

Theorem 1. The number of sets of size k that can be selected from a set of size n is $\binom{n}{k}$.

*Proof.*Let $A(n, k)$ be the number of ways we can select k items from a set of size n. If $k = 0$, then there is one choice to pick the empty set. If $n = 1$ and $k = 1$, then there is one way (pick the single element).

Consider now the general case $n \geq k \geq 1$. Pick an element from the set of size n. Then we have two options: (1) include it in the subset and (ii) do not include it. In case (1), we have to pick $k-1$ items from the remaining elements, i.e. a set of size $n - 1$.

In case (2), we have to pick the remaining $k - 1$ items from the remaining elements, i.e. a set of size $n - 1$. Hence, $A(n, k) = A(n - 1, k - 1) + A(n - 1, k)$ and so it satisfies the recurrence relation of Property 1. As we established that the base cases match $A(n, k) = \binom{n}{k}$. □

This combinatorial viewpoint allows for a variety of arguments involving binomial coefficients. Let's see a simple example.

Property 2. For natural n and k,

$$\binom{n}{k} = \binom{n}{n-k}.$$

*Proof.***(Combinatorial):** The LHS counts the number ways to select k items from a set of size n. But, selecting k items is the same as choosing the $n - k$ items that we will not select. The RHS counts the number of ways to select $n - k$ items from a set of size n, so the two counts must be equal.

(Algebraic): Of course, we can solve this by expansion,

$$\binom{n}{k} = \frac{n!}{(n-k)!k!} = \frac{n!}{(n-k)!(n-(n-k))!} = \binom{n}{n-k}.$$

$\square$

Let's see another example which is a bit more involved.

Property 3. For all natural n,

$$\sum_{k=0}^{n} \binom{n}{k} = 2^n.$$

Proof. **(Combinatorial):** The LHS counts the numbers of ways to make subsets of size 0, size 1, size 2, ..., size n. This is equal to the number of possible subsets of a set of size n. For each of the n elements we have 2 choices: (i) include it or (ii) do not include it. Hence, there must be 2^n possible outcomes. Hence,

$$\sum_{k=0}^{n} \binom{n}{k} = 2^n.$$

(See below - Binomial theorem): An alternative approach to prove this is to use the Binomial theorem for $x = y = 1$. Hence,

$$2^n = (1+1)^n = \sum_{k=0}^{n} \binom{n}{k} 1^i 1^{n-i} = \sum_{k=0}^{n} \binom{n}{k}.$$

$\square$

Property 4. Show that for natural n and k with $n \geq k$,

$$k \cdot \binom{n}{k} = n \binom{n-1}{k-1}.$$

Proof. We start from the RHS,

$$n \binom{n-1}{k-1} = n \cdot \frac{(n-1)!}{(k-1)!(n-k)!} = \frac{n!}{(k-1)!(n-k)!} = k \cdot \binom{n}{k}.$$

$\square$

Exercise 1. Show that for natural numbers $n \geq k \geq r \geq 0$,

$$\binom{n}{k}\binom{k}{r} = \binom{n}{r}\binom{n-r}{k-r}.$$

Exercise 2. For $2 \leq k \leq n-2$ and $n \geq 4$, show that

$$\binom{n}{k} = \binom{n-2}{k-2} + 2\binom{n-2}{k-1} + \binom{n-2}{k}.$$

The binomial theorem

Theorem 2. For any $x, y \in \mathbb{R}$, for natural n,

$$(x+y)^n = \sum_{i=0}^{n} \binom{n}{i} x^i y^{n-i}.$$

Proof. **(By induction):** One way to prove this is by induction on n.

(Combinatorial): Another way to prove it is using a combinatorial argument. We will count the number of times that $x^i y^{n-i}$ appears in the expansion of $(x+y)^n$. Note that each term in the expansion is formed by choosing an x or a y from each $(x+y)$ term,

$$(x+y)^n = \underbrace{(x+y)\cdot(x+y)\cdot\ldots\cdot(x+y)\cdot(x+y)}_{n \text{ terms}}.$$

In order to construct the term $x^i y^{n-i}$, we have to pick i times an x and $n-i$ times a y. There are $\binom{n}{i}$ ways to pick the $(x+y)$ terms were we pick an x, so this is the number of times that $x^i y^{n-i}$ will form (and hence its coefficient). So,

$$(x+y)^n = \sum_{i=0}^{n} \binom{n}{i} x^i y^{n-i}.$$

$\square$

Note: Both of these proofs generalise to the multinomial theorem, which you will see in the Part IA Probability course.

Now, there are various interesting things that we can do with this identity. For example, by choosing $x = y = 1$, we get that

$$2^n = (1+1)^n = \sum_{i=0}^{n} \binom{n}{i} 1^i 1^{n-i} = \sum_{i=0}^{n} \binom{n}{i}.$$

Property 5. For any natural n,

$$\sum_{i=0}^{n} (-1)^i \binom{n}{i} = 0.$$

*Proof.*By choosing $x = 1$ and $y = -1$,

$$0 = ((-1) + 1)^n = \sum_{i=0}^{n} \binom{n}{i} (-1)^i (1)^{n-i} = \sum_{i=0}^{n} \binom{n}{i} (-1)^i.$$

□

This allows us to deduce the following:

Property 6. For any natural $n \geq 1$,

$$\sum_{i \geq 0} \binom{n}{2i} = \sum_{i \geq 0} \binom{n}{2i+1} = 2^{n-1}.$$

*Proof.*We group on one side the negative and on the other side the positive terms of the expression we proved in the previous property to get,

$$\sum_{i=0}^{n} (-1)^i \binom{n}{i} = 0 \Rightarrow \sum_{i \geq 0} \binom{n}{2i} = \sum_{i \geq 0} \binom{n}{2i+1},$$

since $(-1)^{2i} = 1$ and $(-1)^{2i+1} = -1$. As these two are equal and sum to 2^n, each of them must be 2^{n-1}. □

Note that in the Fibonacci handout we wanted to prove that $2^{n-1} \mid \binom{n}{1} + \binom{n}{3} \cdot 5 + \binom{n}{5} \cdot 5^2 + \ldots + + \binom{n}{2i+1} 5^i + \ldots$. How would you show this?

Proof.

$$0 = ((-5)+5)^n = \sum_{i=0}^{n} \binom{n}{i} (-5)^i (5)^{n-i}$$
$$10^n = (5+5)^n = \sum_{i=0}^{n} \binom{n}{i} (5)^i (5)^{n-i}$$

By subtracting the second from the first one, only the odd terms remain on the RHS, so

$$2 \cdot \sum_{i \geq 0} \binom{n}{2i+1} (5)^i (5)^{n-i} = 10^n = 2^n 5^n.$$

By cancelling out the 2, this implies that $2^{n-1} \mid \sum_{i\geq 0} \binom{n}{2i+1} (5)^i (5)^{n-i}$. □

Exercise 3. Show that for natural n,

$$\sum_{i=0}^{n} 2^i \binom{n}{i} = 3^n$$

Property 7. Show that for natural n,

$$\sum_{i=0}^{n} i \cdot \binom{n}{i} = n \cdot 2^{n-1}.$$

*Proof.***(By differentiation):** By the Binomial theorem for $y = 1$, we have $(x+1)^n = \sum_{i=0}^{n} \binom{n}{i} x^i 1^{n-i} = \sum_{i=0}^{n} \binom{n}{i} x^i$, By differentiating both sides with respect to x, we have

$$n(x+1)^{n-1} = \sum_{i=1}^{n} \binom{n}{i} i \cdot x^{i-1}$$

Now by evaluating the derivative at $x = 1$, we get,

$$n2^{n-1} = \sum_{i=1}^{n} i \cdot \binom{n}{i}$$

(Using properties):

$$\begin{aligned}
\sum_{i=0}^{n} i \cdot \binom{n}{i} &= \sum_{i=1}^{n} i \cdot \binom{n}{i} \text{ (since } 0 \cdot \binom{n}{0} = 0) \\
&= \sum_{i=1}^{n} n \cdot \binom{n-1}{i-1} \text{ (By Property 4)} \\
&= n \sum_{i=1}^{n} \binom{n-1}{i-1} \\
&= n \sum_{i=0}^{n-1} \binom{n-1}{i} \text{ (By reindexing)} \\
&= n2^{n-1} \text{ (By property 3)}
\end{aligned}$$

□

By generalising the differentiation technique show that:

Exercise 4. For any natural number $n \geq 2$,

$$\sum_{i=0}^{n} i(i-1) \cdot \binom{n}{i} = n(n-1)2^{n-2}.$$

Exercise 5. For any natural number $n \geq 2$,

$$\sum_{i=0}^{n} i(-1)^{i+1} \cdot \binom{n}{i} = 0.$$

Exercise 6. For any natural number $n \geq 2$,

$$\sum_{i=0}^{n} i^2 \cdot \binom{n}{i} = n(n+1)2^{n-2}.$$

Hint: Use Property 7 and Exercise 4.

Exercise 7. By integration, show that

$$\sum_{i=0}^{n} \frac{1}{i+1} \binom{n}{i} = \frac{2^{n+1}}{n+1}.$$

Exercise 8. By considering $(1+x)^n(1+x)^n = (1+x)^{2n}$ show that

$$\sum_{i=0}^{n} \binom{n}{i}^2 = \binom{2n}{n}.$$

See this article for more solutions.

Property 8 (Chu-Vandermonde)**.** Show that for naturals n, m, r with $r \leq m+n$.

$$\binom{n+m}{r} = \sum_{k=0}^{r} \binom{m}{k} \cdot \binom{n}{r-k}.$$

Proof. **(Binomial Theorem)** *Hint:* Consider the r-th term in the expressions $(1+x)^{n+m}$ and $(1+x)^n(1+x)^m$.
(Combinatorial) *Hint:* Consider the number of ways to make a committee of r items out of n women and m men. We can pick $r = 0, 1, \dots, n$ women in the committee. ☐

More exercises

Exercise 9. Prove using induction that for naturals n, r

$$\sum_{k=0}^{n} \binom{r+k}{k} = \binom{r+n+1}{n}.$$

See answer here.

Exercise 10. Prove that for naturals n and k with $k \leq n$.

$$\binom{k}{k} + \binom{k+1}{k} + \ldots + \binom{n}{k} = \binom{n+1}{k+1}.$$

Hint: For a combinatorial proof think about counting the number of subsets of size k where the largest element is $k+i$.

Exercise 11. (a) Show that $\sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n-k}{k} = F_{n+1}$

(b) Where can you find the Fibonacci numbers in Pascal's triangle?

Exercise 12. (a) For $n \geq 2$, prove that

$$\binom{2}{2} + \binom{3}{2} + \ldots + \binom{n}{2} = \binom{n+1}{3}.$$

(b) By observing that $m^2 = 2\binom{m}{2} + m$ for $m \geq 2$ deduce the sum of squares formula,

$$1^2 + 2^2 + \ldots + n^2 = \frac{n(n+1)(2n+1)}{6}.$$

(c) By applying part (a), show that

$$1 \cdot 2 + 2 \cdot 3 + \ldots + n \cdot (n+1) = \frac{n(n+1)(n+2)}{3}.$$

Exercise 13. (a) Show that

$$\binom{2n}{n} = \frac{1 \cdot 3 \cdot 5 \cdot \ldots \cdot (2n-1)}{2 \cdot 4 \cdot 6 \cdot \ldots \cdot (2n)}.$$

(b) Establish the inequality for $n \geq 1$,

$$2^n < \binom{2n}{n} < 2^{2n}.$$

Past papers

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(*b*) Recall that, for $i, j \in \mathbb{N}$,

$$\binom{i}{j} \stackrel{\text{def}}{=} \begin{cases} 0 & \text{, if } i < j \\ \frac{i!}{j!(i-j)!} & \text{, if } i \geq j \end{cases}$$

(*i*) Show that for all $m < l$ in $\mathbb{N}$,

$$\binom{l}{m+1} + \binom{l}{m} = \binom{l+1}{m+1}$$

[2 marks]

(*ii*) Prove that

$$\forall n \in \mathbb{N}.\ \forall m \in \mathbb{N}.\ 0 \leq m \leq n \implies \sum_{k=0}^{n} \binom{k}{m} = \binom{n+1}{m+1}$$

[6 marks]

Further references

There are a number of things that this handout did not cover.

- The generalisation of the binomial coefficients for $k > n$. This is for example used in the Negative binomial distribution and various Taylor expansions.
- The generating functions approach and the finite calculus approach for deriving identities. (See Christmas projects)

- The generalisation of the binomial coefficients for real numbers via the Γ function.
- You may be interested in the Fibonacci-Binomial coefficients (see here).