

Algorithms Example Sheet 5: Core Questions

Minimum Spanning Trees

Exercise 5.C.1 [Basic tree properties] In this exercise, you will prove some of the basic properties about trees that are useful in analysing spanning trees:

- (a) Show that adding an edge to a tree creates a unique cycle.
- (b) Show that given a tree with a cycle, removing any edge from the cycle, leaves us with a tree.
- (c) Given an undirected cycle C where each vertex is coloured either red or blue. Given that there is an edge (u, v) with $u \in R$ and $v \in B$, show that there must be another edge $e' = (u', v')$ on the cycle such that $u' \in R$ and $v' \in B$.
- (d) Deduce that given a cycle C and a cut $(S, V \setminus S)$ in an undirected graph G , if there is one edge e crossing the cut, then there must be another e' that also crosses the cut.

Exercise 5.C.2 [Basic properties]

- (a) Define the *minimum spanning tree* of a weighted undirected graph.
- (b) **[Cut property]** Consider a weighted undirected graph G and a cut S . There exists an MST that contains the edge e with lightest weight in the cut.
- (c) What happens if e has strictly the lightest weight in the cut? What can you say about the heaviest edge in the cut?
- (d) **[Cycle property]** Consider a weighted undirected graph G and an edge e . If e is greater than all other edges in a cycle, then it cannot belong to any MST of G .
- (e) What happens if there are multiple heaviest edges? What can you say about the lightest edge?
- (f) In a connected undirected graph with edge weights ≥ 0 , let $u \leftrightarrow v$ be a minimum-weight edge. Show that $u \leftrightarrow v$ belongs to a minimum spanning tree.

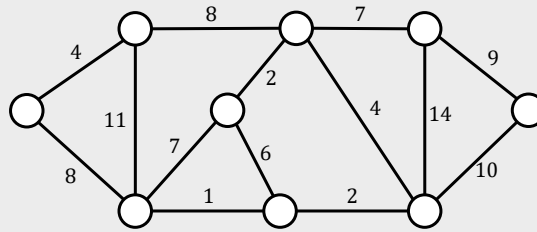
[Exercise 16 in Lecturer's handout]

Exercise 5.C.3 [Prim's algorithm]

- (a) Define Prim's algorithm.
- (b) Write out a formal proof of correctness of Prim's algorithm. You may use without proof the theorem stated in lecture notes: "Suppose we have a forest F and a cut C such that (i) C contains no edges of F , and (ii) there exists a MST containing F . If we add to F a min-weight edge among those that cross C , then the result is still part of a MST." [Note: the proof of this theorem is not examinable, but the application to Prim's algorithm is examinable. Also note that the term cut has two slightly different definitions, one for flow networks, one for spanning trees.]

[Exercise 10 in Lecturer's handout]

- (c) What is the time complexity of the implementation of Prim's algorithm?
- (d) Describe how Prim operates on the following graph.

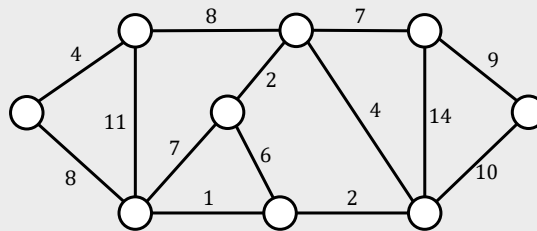


[Exercise 8 in Lecturer's handout]

- (e) Implement Prim's algorithm.

Exercise 5.C.4 [Kruskal's algorithm]

- (a) Define Kruskal's algorithm.
- (b) Write out a formal proof of correctness of Kruskal's algorithm.
- (c) What is the time complexity of the implementation of Kruskal's algorithm?
- (d) Describe how Kruskal operates on the following graph.

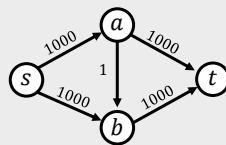


[Exercise 8 in Lecturer's handout]

- (e) Implement Kruskal's algorithm.
- (f) Are there any cases in which you would prefer Kruskal's algorithm?

Maximum flow

Exercise 5.C.5 Use the Ford-Fulkerson algorithm, by hand, to find the maximum flow from s to t in the following graph. How many iterations did you take? What is the largest number of iterations it might take, with unfortunate choice of augmenting path?



Exercise 5.C.6 [Non-Terminating example for Ford-Fulkerson] (optional) Look at the example graph presented [here](#) (and [here](#) or [here](#)).

- (a) Prove that r (usually denoted as ψ , being related to the golden ratio ϕ) satisfies $r^2 = 1 - r$.
- (b) What is the maximum flow for the given graph?
- (c) Define formally the pathological sequence of augmented paths.
- (d) Does this sequence converge to the maximum flow? What does this mean?

Exercise 5.C.7 Argue why the Ford-Fulkerson algorithm terminates when the edge capacities are rational (in a finite graph).

Exercise 5.C.8 Consider a flow f on a directed graph with source vertex s and sink vertex t . Let $f(u \rightarrow v)$ be the flow on edge $u \rightarrow v$, and set $f(u \rightarrow v) = 0$ if there is no such edge.

(a) Show that

$$\sum_{v \neq s, t} \left(\sum_w f(v \rightarrow w) - \sum_u f(u \rightarrow v) \right) = 0.$$

(b) The value of the flow is defined to be the net flow out of s ,

$$\text{value}(f) = \sum_w f(s \rightarrow w) - \sum_u f(u \rightarrow s).$$

Prove that this is equal to the net flow into t . [*Hint*: Add the LHS of the equation from part (a)]

Exercise 5.C.9 The code for `ford_fulkerson` as given in the handout has a bug: lines 27-39, which augment the flow, rely on an unstated assumption about the augmenting path. Give an example which makes the code fail. State the required assumption, and prove that the assertion on line 39 is correct, i.e. that after augmenting we still have a valid flow.