Balanced Allocations: Relaxing Two-Choice

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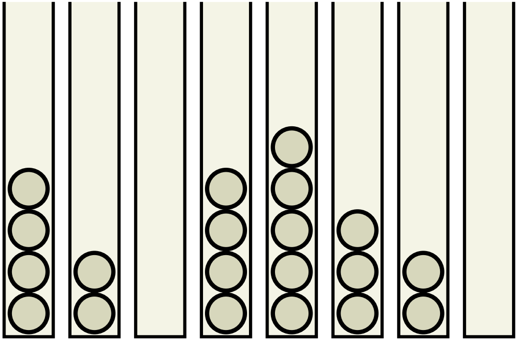

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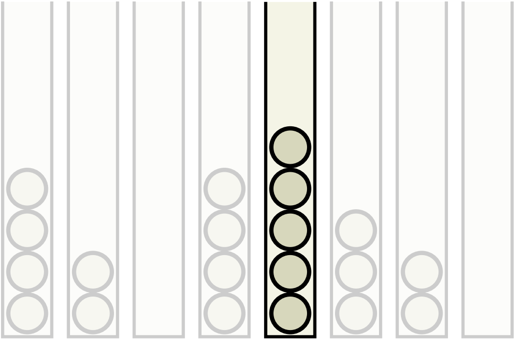

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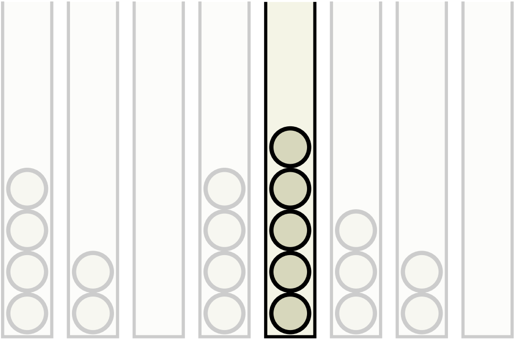

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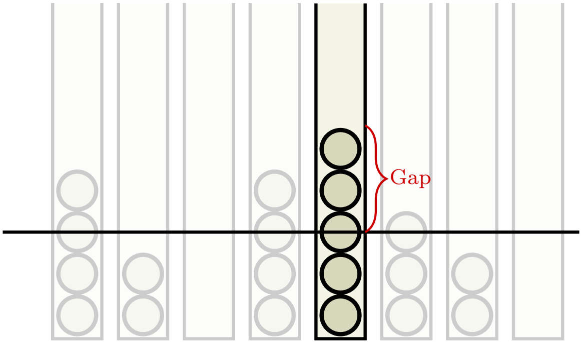

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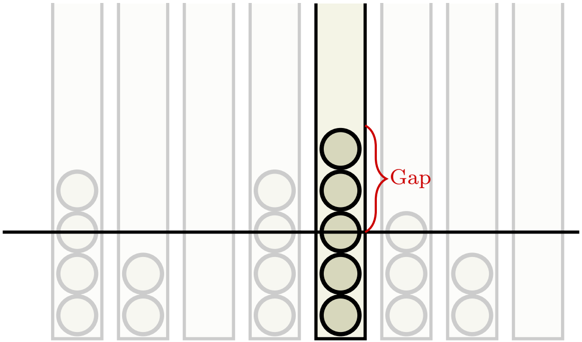

- Applications in hashing, load balancing and routing.

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Meaning with probability at least $1 - n^{-c}$ for constant $c > 0$.

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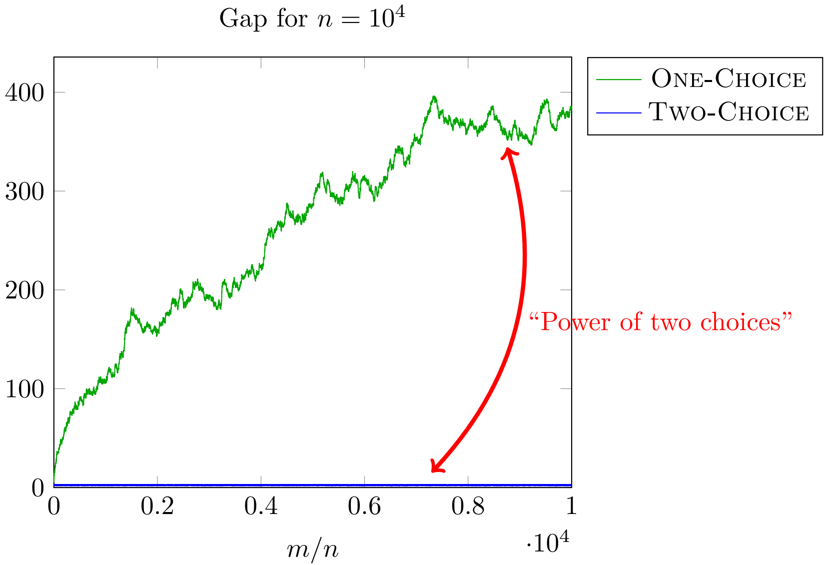

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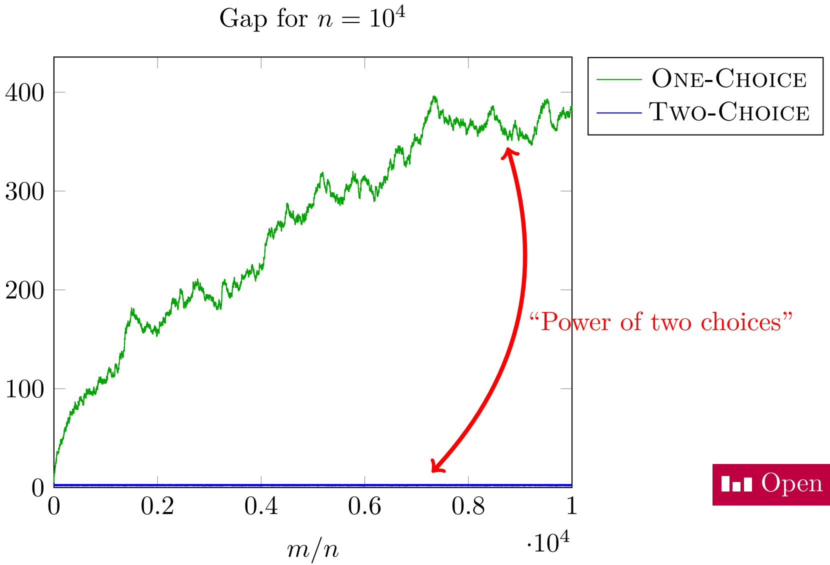

Open in Visualiser.

Relaxing with incomplete information

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Iteration: For each $t \geq 0$, sample two bins i_1 and i_2 u.a.r., and update:

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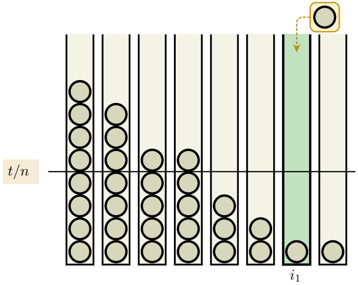

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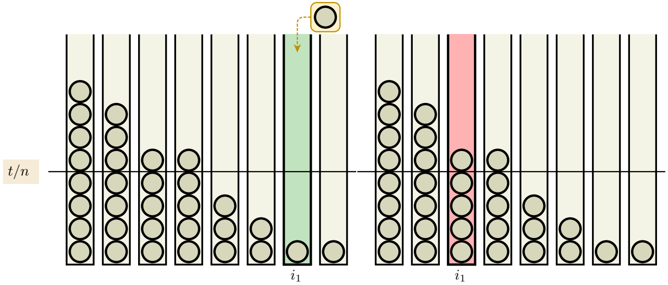

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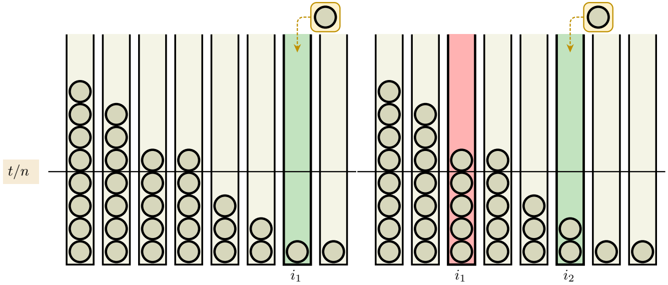

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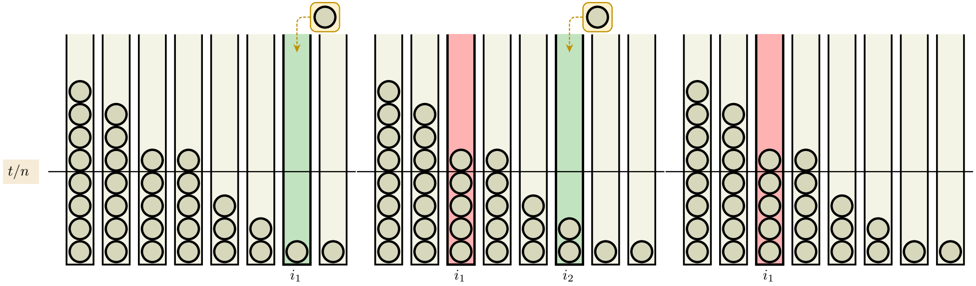

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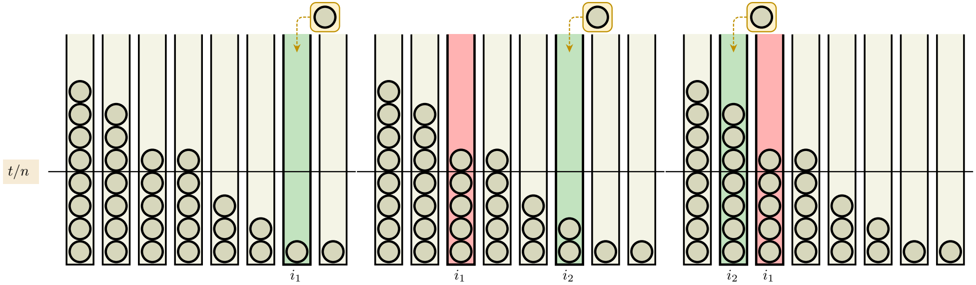

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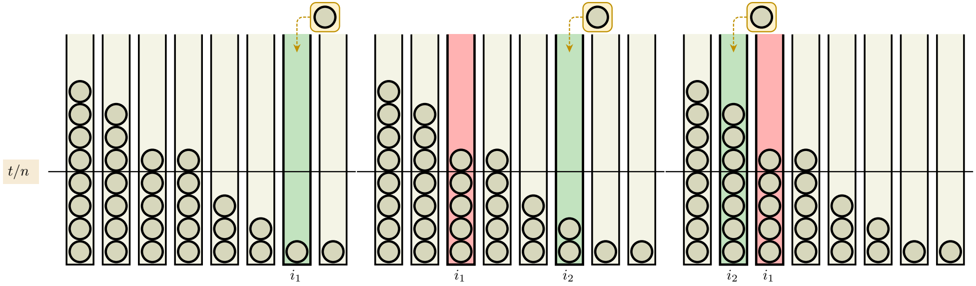

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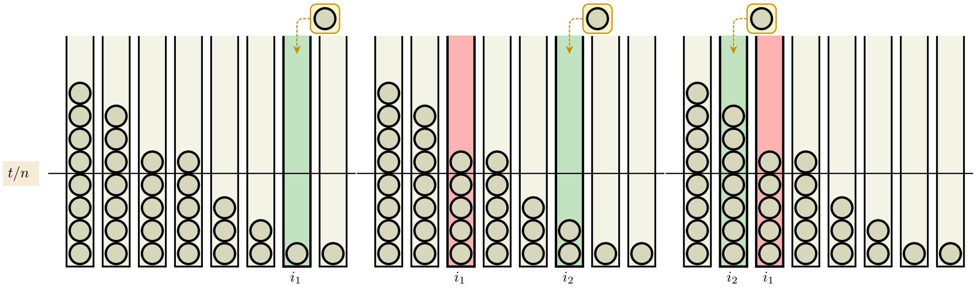

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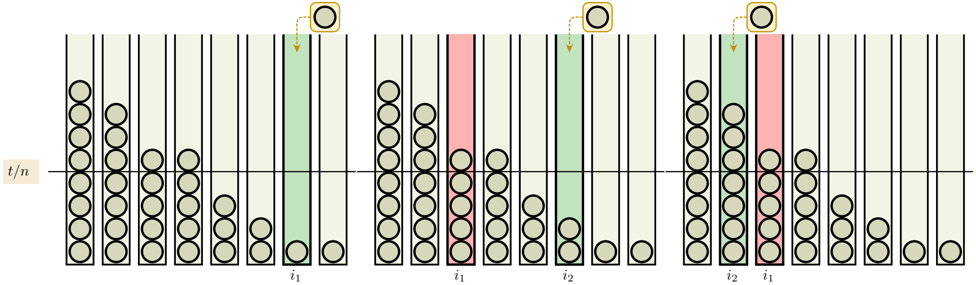

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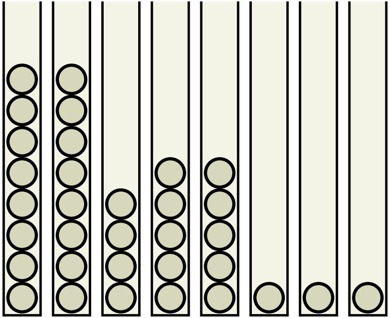

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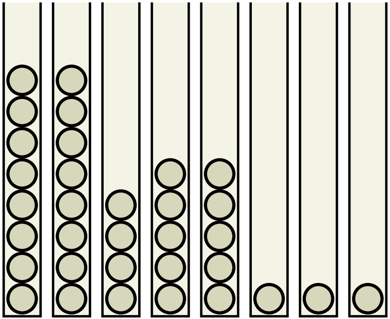

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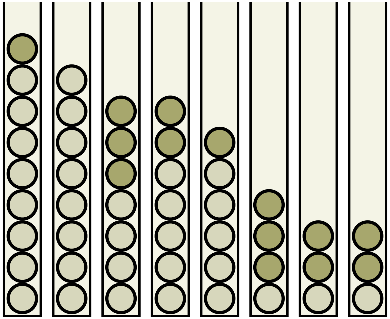

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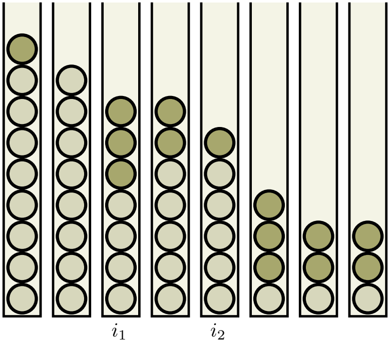

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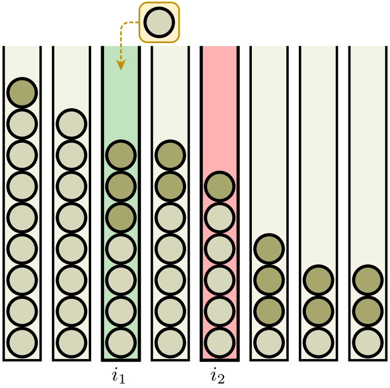

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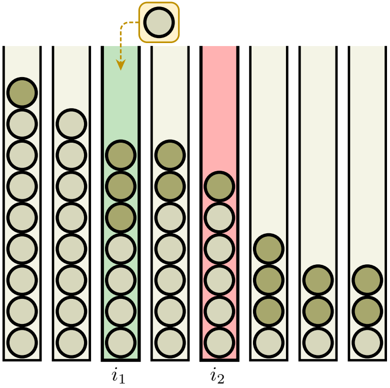

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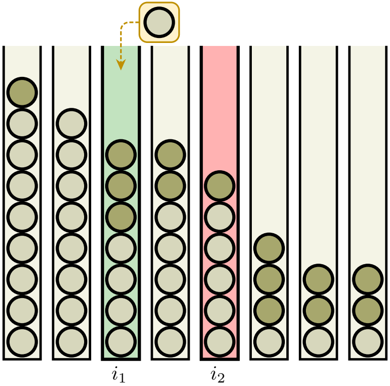

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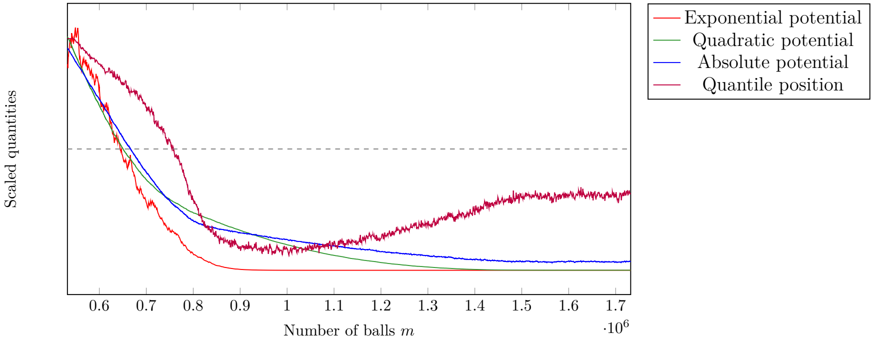
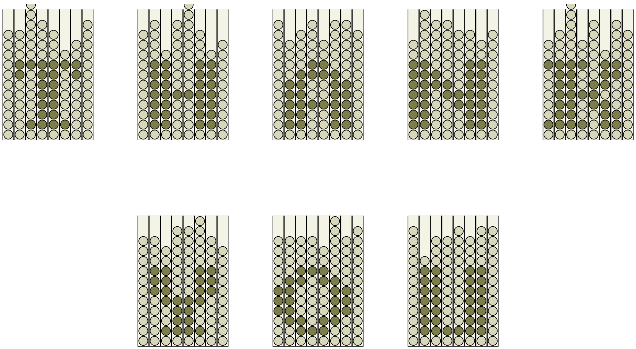
- For $b \geq n \log n$, achieves $\text{Gap}(m) = \Theta(b/n)$.

Our techniques

- Interplay between (i) **linear**, (ii) **quadratic** and (iii) **exponential** potentials.





Visualisations: `dimitrioslos.com/halg22`

Bibliography I

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Bibliography II