Naively sorting evolving data is optimal and robust

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Background

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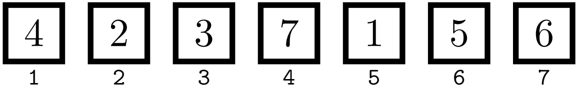

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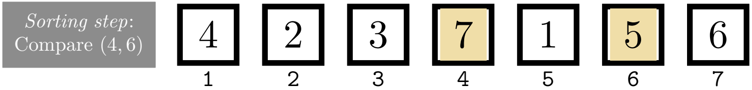

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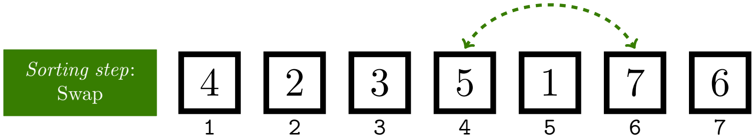

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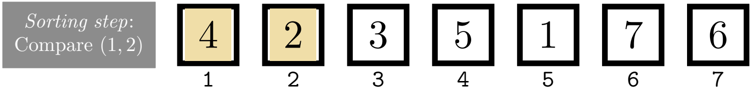

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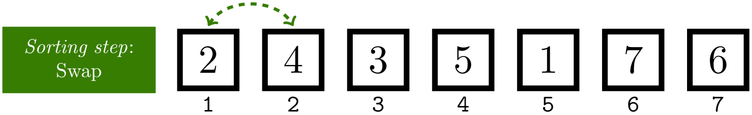

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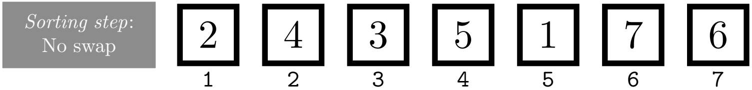

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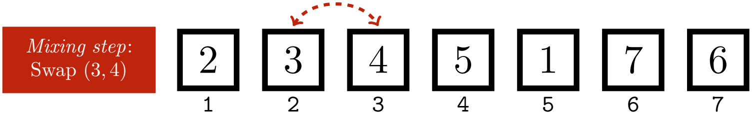

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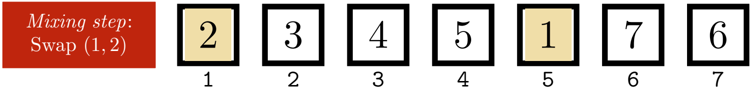

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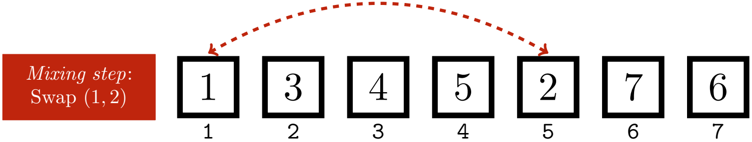

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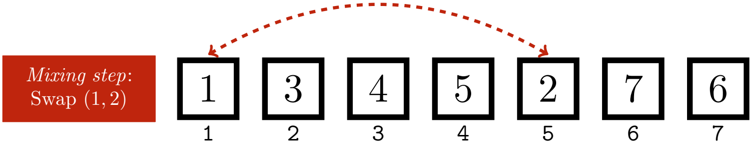

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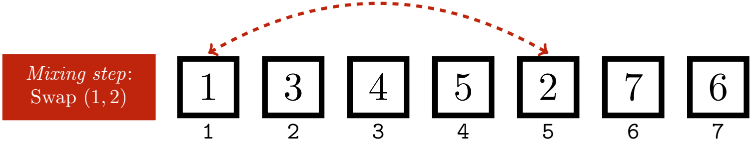

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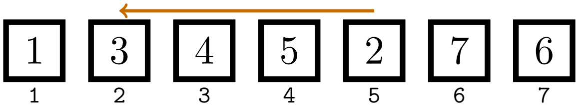

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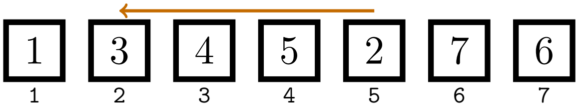

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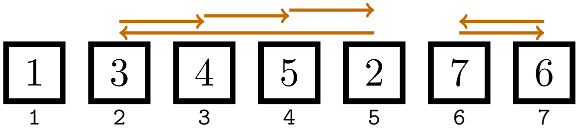

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- Mahdian [Mah14] proposed Naive-Sort which in each step samples a pair of *adjacent* items; and sort them if they are out of order.

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  - A more *refined version* of QuickSort achieves w.h.p. $\mathrm{tdev} = O(n \log \log n)$.
  - For any sorting algorithm, $\mathrm{tdev} = \Omega(n)$.
  - *Conjecture:* For any const $b \geq 1$, there exists an algorithms which matches this.

- Besa Vial, Devanny, Eppstein, Goodrich and Johnson [BVDE+18a] showed that Insertion-Sort for $b = 1$ achieves w.h.p. $\mathrm{tdev} = O(n)$ in $O(n^2)$ steps.

  $\rightsquigarrow$ Quadratic time algorithms are optimal.
  - Interleaved with QuickSort converges in $O(n \log n)$ steps.
  - Proof tailored for $b = 1$. *Open problem:* What happens for $b > 1$?
  - Besa Vial et al. [BVDE+18b] *observed* bounds extend to more powerful mixing steps (e.g. *hot-spot adversary*)?

- Mahdian [Mah14] proposed Naive-Sort which in each step samples a pair of *adjacent* items; and sort them if they are out of order.
  - *Open problem:* Does Naive-Sort achieve the optimal mdev and tdev?

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- To preserve increasing distances needed for the sorting analysis, we require that

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- **Main idea:** Maintain a *target array* τ_t (initially $\tau_0 = \texttt{id}_n$)

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Part B: The $\mathrm{tdev} = O(n)$ bound

Why the previous analysis does not extend

- The previous analysis uses *long* intervals of length kn.
- **Problem:** When $k = o(\log n)$, then $\mathrm{mdev}(\tau) = \omega(k)$.
- For example, when $k = \Theta(1)$, it follows that $\mathrm{mdev}(\tau) = \Theta(\log n / \log\log n)$, and so the previous upper bound *does not* guarantee a decrease

$$\mathbf{E}\left[\,\Phi_{t+kn} | \Phi_t\,\right] \leq n \cdot e^{-\Theta(1)} \cdot e^{\Theta(\log n/\log\log n)} = n \cdot e^{\Theta(\log n/\log\log n)}.$$

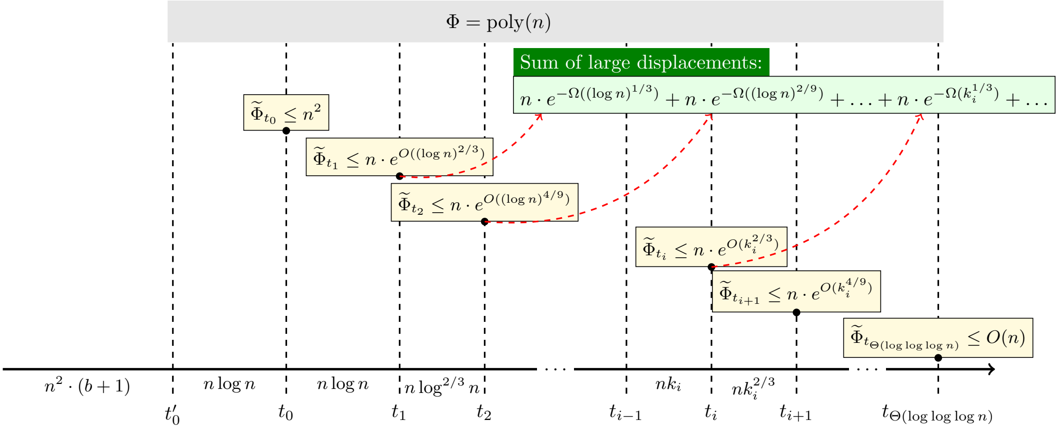
- **Obsrvation:** We are assuming worst-case bound on displacement for *all* items in τ.
- **Solution:** *Reset* targets only for items with small displacements (those below $O(k^{2/3})$).
  - *Small displacements*: handle as in previous analysis.
  - *Large displacements*: show that on aggregate they don't contribute much.
- We prove a concentration inequality (c.f. [LS22]) showing that

$$\mathbf{Pr}\left[\Phi_t \leq n \cdot e^{O(k^{2/3})}\right] \geq 1 - n^{-2}.$$

which implies that *aggregate* contribution of large displacements is $n \cdot e^{-\Omega(k^{1/3})}$.

Putting it all together

- We repeat for $\Theta(\log\log\log n)$ iterations.
- In iteration i, we set $k_{i+1} = (k_i)^{2/3}$.



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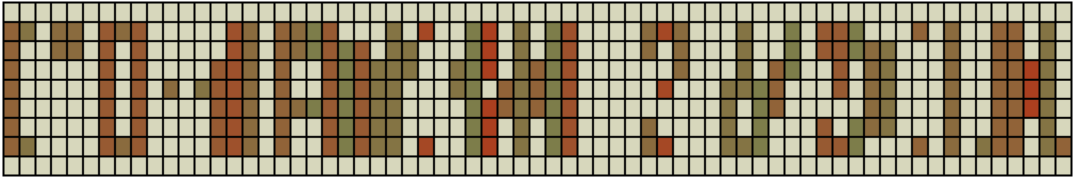
- Apply *analysis techniques* to related problems.



For more visualisations, see: `team.inria.fr/wide/papers/focs24`

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